Hilbert Space Representations of Cross Product Algebras

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Abstract: Hilbert space representations of cross product ∗-algebras of the Hopf ∗-algebras $\mathcal{U}_q(gl_2)$ with the coordinate algebras $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ and $\mathcal{O}(\mathbb{R}_q^3)$ of quantum vector spaces, and of $\mathcal{U}_q(su_2)$ with the coordinate algebras $\mathcal{O}(SU_q(2))$ and $\mathcal{O}(S_q^2)$ of the corresponding quantum spheres, are investigated and classified. Invariant states on the coordinate ∗-algebras are described by two variants of the quantum trace.

1   Introduction

Let $\mathcal{U}$ be a Hopf ∗-algebra and let $\mathcal{X}$ be a unital right $\mathcal{U}$-module ∗-algebra. Then the smash product algebra $\mathcal{U}\#\mathcal{X}$ [M] is a ∗-algebra which contains $\mathcal{U}$ and $\mathcal{X}$ as ∗-subalgebras. Following the terminology used by operator algebraists we prefer to call the algebra $\mathcal{U}\#\mathcal{X}$ a cross product algebra and denote it by $\mathcal{U}\ltimes\mathcal{X}$. The cross product ∗-algebra has a natural physical interpretation: We think of $\mathcal{X}$ as an algebra of functions on a "quantum space" on which the elements of $\mathcal{U}$ act as "generalized differential operators". Then $\mathcal{U}\ltimes\mathcal{X}$ can be considered as an algebra of differential operators with coefficients in $\mathcal{X}$ and so as a phase space algebra associated with the quantum space. Therefore, as is in usual quantum mechanics, Hilbert space representations of the phase space ∗-algebra $\mathcal{U}\ltimes\mathcal{X}$ play a crucial role in the study of the quantum space.

Suppose there exists a $\mathcal{U}$-invariant state h on $\mathcal{X}$ and let π_h be the GNS-representation of h. Then there is a unique closed ∗-representation of $\mathcal{U}\ltimes\mathcal{X}$, called Heisenberg representation and denoted also by π_h, such that its restriction to $\mathcal{X}$ is the GNS-representation π_h and the cyclic vector v_h is $\mathcal{U}$-invariant.

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In this paper we are concerned with cross product algebras of the Hopf $*$-algebras $\mathcal{U}_q(gl_2)$ acting on the coordinate algebras $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ and $\mathcal{O}(\mathbb{R}_q^3)$ of quantum vector spaces, and of $\mathcal{U}_q(su_2)$ acting on the coordinate algebras $\mathcal{O}(SU_q(2))$ and $\mathcal{O}(S_q^2)$ of the corresponding quantum unit spheres. (Precise definitions are given in Section 3.) The main purpose of this paper is to study Hilbert space representations of these cross product $*$-algebras and to use them to describe invariant states on the coordinate $*$-algebras.

Rrepresentations of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$ has been constructed first in [F] and then in [CW] by using other methods. We rediscover these representations in Subsection 6.4.

There are two principal ways to find and to classify Hilbert space representations of the $*$-algebra $\mathcal{U} \ltimes \mathcal{X}$. We discuss these methods and some of their applications in the case $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$. In the first approach we assume that the restriction $\pi_{\mathcal{U}}$ of a $*$-representation of $\mathcal{U} \ltimes \mathcal{X}$ to the $*$-subalgebra $\mathcal{U}$ is a "well-behaved" representation of $\mathcal{U}$. For $\mathcal{U}_q(su_2)$ we require that $\pi_{\mathcal{U}}$ can be expressed as a direct sum of spin l representations T_l with arbitrary multiplicities. Such $*$-representations of $\mathcal{U}_q(su_2)$ are called integrable. For the Heisenberg representation π_h of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ the representation $\pi_{\mathcal{U}}$ of $\mathcal{U}_q(su_2)$ is integrable and the Peter-Weyl theorem of $\mathcal{O}(SU_q(2))$ gives an explicit decomposition of $\pi_{\mathcal{U}}$ into a direct sum of representations T_l (see e.g. formula (45) on p. 110 in [KS]). We prove (Theorem 5.5) that any closed $*$-representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ obtained from an integrable representation $\pi_{\mathcal{U}}$ of $\mathcal{U}_q(su_2)$ is a direct sum of copies of the Heisenberg representation π_h. In particular, π_h is the only closed irreducible $*$-representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ such that its restriction to $\mathcal{U}_q(su_2)$ is integrable. The Heisenberg representation π_h is used to describe the Haar state h of $\mathcal{O}(SU_q(2))$ as a quantum trace (Theorem 5.7). More precisely, if C_q is the Casimir element and K is the group-like generator of $\mathcal{U}_q(su_2)$, then there exists a holomorphic function $\zeta(z)$ on $z \in \mathbb{C}$, Re $z > 1$, such that

$$h(x) = \zeta(z)^{-1} \operatorname{Tr} \overline{\pi_h(C_q)^{-z}\pi_h(K^{-2})} \, \overline{\pi_h(x)}, \ x \in \mathcal{O}(SU_q(2)).$$

In the second approach we start with a $*$-representation of $\mathcal{X}$ written in a canonical form and we try to extend it to a $*$-representation of $\mathcal{U} \ltimes \mathcal{X}$. Since elements of $\mathcal{U}$ act as unbounded operators, during this derivation we have to add some regularity assumptions concerning the unbounded operators. In the case of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ we end up with irreducible representations $(I)_{H,\epsilon}$ parameterized by numbers $\epsilon \in \{1, -1\}$ and $H \in (q^{1/2}, 1]$. We prove (Theorem

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6.2) that (after extending the domain) the representation $(I)_{1,1}$ is unitarily equivalent to the Heisenberg representation π_h. We use the representation $(I)_{H,\epsilon}$ to express the Haar state h as a partial quantum trace (Theorem 6.4).

Each of two approaches has its advantage: In the first one the generators of $\mathcal{U}_q(su_2)$ act in the standard form used in representation theory, while in the second one the action of the generators of $\mathcal{O}(SU_q(2))$ is given in a canonical form. The second way yields a larger class of representations $(I)_{H,\epsilon}$, because the restriction of $(I)_{H,\epsilon}$ to $\mathcal{U}_q(su_2)$ is not integrable if $H \in (q^{1/2}, 1)$.

This paper is organized as follows. In Section 2 we collect definitions and general facts on cross product $*$-algebras. In Section 3 we define the cross product algebras studied in this paper and characterize them in terms of generators and defining relations. Section 4 contains a number of preliminaries on representations. In Subsection 4.2 we list the canonical form of representations of the coordinate $*$-algebras $\hat{\mathcal{O}}(\mathbb{C}_q^2)$, $\mathcal{O}(\mathbb{R}_q^3)$, $\mathcal{O}(SU_q(2))$, and $\mathcal{O}(S_q^2)$ as used in Section 6. The main results of the paper are contained in Sections 5 and 6. Section 5 is concerned with invariant functionals on the four coordinate $*$-algebras and with Heisenberg representations of the cross product algebras. While Section 5 deals with the first approach as explained above, the second approach is developed in Section 6. For the cross product algebras above and another cross product algebra related to the 3D-calculus on $SU_q(2)$ the representations are listed by explicit formulas of the generators.

Background material on quantum groups can be found in [FRT], [KS], [M], on unbounded representations in [OS], [P], [S] and on the quantum $SU(2)$ in [KS], [VS], [Wo]. Let us collect some definitions and notations used in what follows. The comultiplication, the counit and the antipode of a Hopf algebra are denoted by Δ, ε, and S, respectively. For a coaction φ and the comultiplication Δ we freely use the Sweedler notations $\varphi(x) = x_{(1)} \otimes x_{(2)}$ and $\Delta(x) = x_{(1)} \otimes x_{(2)}$. If T is an operator on a Hilbert space, we denote by $\mathcal{D}(T)$ the domain, by $\sigma(T)$ the spectrum, by $\overline{T}$ the closure and by T^* the adjoint of T. A self-adjoint operator A is called strictly positive if $A \geq 0$ and ker $A = \{0\}$. We write $\sigma(A) \sqsubseteq (a, b]$ if $\sigma(A) \subseteq [a, b]$ and a is not an eigenvalue of A. We say that two self-adjoint operators strongly commute if their spectral projections mutually commute. For a vector η of a Hilbert space $\mathcal{H}_0$, we denote by η_n the vector of $\mathcal{H} = \oplus_{k=0}^{\infty} \mathcal{H}_k$, $\mathcal{H}_k := \mathcal{H}_0$, which has η as its n-th component and zero otherwise and we put $\eta_{-1} := 0$. By a $*$-representation of a $*$-algebra $\mathcal{X}$ on a dense domain $\mathcal{D}$ of a Hilbert space we mean a homomorphism π of $\mathcal{X}$ into the algebra $L(\mathcal{D})$ of linear operators mapping $\mathcal{D}$ into itself such that $\langle \pi(x)\eta, \xi \rangle = \langle \eta, \pi(x^*)\xi \rangle$ for $x \in \mathcal{X}, \eta, \xi \in \mathcal{D}$. A $*$-representation π is called

closed if $\mathcal{D}$ is the intersection of domains $\mathcal{D}(\overline{\pi(x)})$, $x \in \mathcal{X}$.

Throughout this paper we suppose that $q, p \in \mathbb{C}, p \neq 0$, and $q \neq 0, 1, -1$, and we abbreviate

$$\lambda := q - q^{-1}, \gamma := (q + q^{-1})^{1/2}, \lambda_n := (1 - q^{2n})^{1/2}, [k]_q := \lambda^{-1}(q^k - q^{-k}).$$

In Sections 5 and 6 we assume that $0 < q < 1$, $p > 0$, and $p \neq 1$.

2 Cross product algebras: general concepts

Throughout this section we suppose that $\mathcal{U}$ is a Hopf algebra with invertible antipode and that $\mathcal{X}$ is an algebra (without unit in general).

Let $\mathcal{X}$ be a right $\mathcal{U}$-module algebra, that is, $\mathcal{X}$ is a right $\mathcal{U}$-module with action $\triangleleft$ satisfying

$$(xy) \triangleleft f = (x \triangleleft f_{(1)})(y \triangleleft f_{(2)}), \;\; x, y \in \mathcal{X}, f \in \mathcal{U} \; .$$

Then the vector space $\mathcal{U} \otimes \mathcal{X}$ is an algebra, called a *right cross product algebra* and denoted by $\mathcal{U} \ltimes \mathcal{X}$, with product defined by

$$(g \otimes x)(f \otimes y) = g f_{(1)} \otimes (x \triangleleft f_{(2)}) y, \;\; x, y \in \mathcal{X}, g, f \in \mathcal{U} \; . \tag{1}$$

Let $\mathcal{U}_0$ be a subalgebra and a right coideal (that is, $\Delta(\mathcal{U}_0) \subseteq \mathcal{U}_0 \otimes \mathcal{U}$) of the Hopf algebra $\mathcal{U}$. By (1), the subspace $\mathcal{U}_0 \otimes \mathcal{X}$ of $\mathcal{U} \otimes \mathcal{X}$ is a subalgebra of $\mathcal{U} \ltimes \mathcal{X}$ which we denote by $\mathcal{U}_0 \ltimes \mathcal{X}$. The importance of such algebras stems from the fact that the unital subalgebra $\mathcal{U}_0$ generated by the quantum tangent space of a left-covariant differential calculus on $\mathcal{A}$ is a right coideal of the Hopf dual $\mathcal{A}^\circ$ ([KS], Proposition 14.5). Since $\mathcal{A}$ is a right $\mathcal{A}^\circ$-module algebra (with action given by formula (9) below), the algebra $\mathcal{U}_0 \ltimes \mathcal{A}$ is well defined.

If $\mathcal{X}$ has also a unit element, then we can consider $\mathcal{X}$ and $\mathcal{U}_0$ as subalgebras of $\mathcal{U}_0 \ltimes \mathcal{X}$ by identifying $f \otimes 1$ with f and $1 \otimes x$ with x. Then $\mathcal{U}_0 \ltimes \mathcal{X}$ is the algebra generated by the two subalgebras $\mathcal{U}_0$ and $\mathcal{X}$ with respect to the cross commutation relation

$$xf = f_{(1)}(x \triangleleft f_{(2)}), \;\; x \in \mathcal{X}, f \in \mathcal{U}_0 \; , \tag{2}$$

or equivalently,

$$fx = (x \triangleleft S^{-1}(f_{(2)})) f_{(1)}, \;\; x \in \mathcal{X}, f \in \mathcal{U}_0 \; . \tag{3}$$

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Inside the algebra $\mathcal{U} \ltimes \mathcal{X}$ the right action $\lhd$ of $\mathcal{U}$ on $\mathcal{X}$ can be nicely expressed by the right adjoint action of the Hopf algebra $\mathcal{U}$. Recall that for any $\mathcal{U}$-bimodule M the right adjoint action

$$ad_R(f)m = S(f_{(1)})mf_{(2)}, \ f \in \mathcal{U}, m \in M,$$

is a well defined right action of $\mathcal{U}$ on M. The subalgebra $\mathcal{X}$ of $\mathcal{U} \ltimes \mathcal{X}$ is obviously a $\mathcal{U}$-bimodule. Using (2) we compute

$$ad_R(f)x{=}S(f_{(1)})xf_{(2)}{=}S(f_{(1)})f_{(2)}(x{\lhd}f_{(3)}) = \varepsilon(f_{(1)})(x{\lhd}f_{(2)}) = x{\lhd}f \ . \qquad (4)$$

Let us turn now to $*$-structures. Suppose that $\mathcal{U}$ is a Hopf $*$-algebra and $\mathcal{X}$ is a right $\mathcal{U}$-module $*$-algebra. The latter means that $\mathcal{X}$ is a right module algebra and a $*$-algebra such that the right action $\lhd$ and the involution $*$ satisfy the compatibility condition

$$(x{\lhd}f)^* = x^*{\lhd}S(f)^*, \ x \in \mathcal{X}, f \in \mathcal{U} \ . \qquad (5)$$

Lemma 2.1 *Let $\mathcal{U}_0$ be $*$-subalgebra and a right coideal of the Hopf $*$-algebra $\mathcal{U}$. Then the algebra $\mathcal{U}_0 \ltimes \mathcal{X}$ is a $*$-algebra with involution given by*

$$(f \otimes x)^* := f_{(1)}^* \otimes (x^*{\lhd}f_{(2)}^*), f \in \mathcal{U}_0, x \in \mathcal{X} \ . \qquad (6)$$

Proof. In order to prove that this defines an algebra involution we use the formulas (1) and (5) and compute

$$
\begin{aligned}
((g \otimes x)(f \otimes y))^* &= (gf_{(1)} \otimes (x{\lhd}f_{(2)})y)^* \\
&= f_{(1)}^* g_{(1)}^* \otimes (y^*(x{\lhd}f_{(3)})^*){\lhd}(f_{(2)}^* g_{(2)}^*) \\
&= f_{(1)}^* g_{(1)}^* \otimes (y^*{\lhd}(f_{(2)}^* g_{(2)}^*))(x^*{\lhd}(S(f_{(4)})^* f_{(3)}^* g_{(3)}^*)) \\
&= f_{(1)}^* g_{(1)}^* \otimes ((y^*{\lhd}f_{(2)}^*){\lhd}g_{(2)}^*)(x^*{\lhd}g_{(3)}^*) \\
&= (f_{(1)}^* \otimes y^*{\lhd}f_{(2)}^*)(g_{(1)}^* \otimes x^*{\lhd}g_{(2)}^*) \\
&= (f \otimes y)^*(g \otimes x)^* \ , \\
(f \otimes x)^{**} &= (f_{(1)}^* \otimes x^*{\lhd}f_{(2)}^*)^* = f_{(1)} \otimes (x^*{\lhd}f_{(3)}^*)^*{\lhd}f_{(2)} \\
&= f_{(1)} \otimes x{\lhd}(S^{-1}(f_{(3)})f_{(2)}) = f \otimes x \ . \qquad \square
\end{aligned}
$$

If $\mathcal{X}$ has a unit, then the involution (6) reads $(f \otimes x)^* = (1 \otimes x)^*(f \otimes 1)^*$.

The above definitions and facts have their left handed counterparts. Suppose that $\mathcal{X}$ is a left module algebra of a Hopf algebra $\mathcal{U}$ with left action $\rhd$.

Then the vector space $\mathcal{X} \otimes \mathcal{U}$ is an algebra, called a *left cross product algebra* and denoted by $\mathcal{X} \rtimes \mathcal{U}$, with product defined by

$$(y \otimes f)(x \otimes g) = y(f_{(1)} \triangleright x) \otimes f_{(2)} g, \ x, y \in \mathcal{X}, f, g \in \mathcal{U} .$$

If $\mathcal{U}_0$ is a subalgebra of $\mathcal{U}$ which is a left coideal (i.e. $\Delta(\mathcal{U}_0) \subseteq \mathcal{U} \otimes \mathcal{U}_0$), then the subspace $\mathcal{X} \otimes \mathcal{U}_0$ is a subalgebra of $\mathcal{X} \rtimes \mathcal{U}$ which is denoted by $\mathcal{X} \rtimes \mathcal{U}_0$. If $\mathcal{X}$ has a unit, then $\mathcal{X} \rtimes \mathcal{U}_0$ can be considered as the algebra generated by the subalgebras $\mathcal{X}$ and $\mathcal{U}_0$ with cross relation

$$fx = (f_{(1)} \triangleright x) f_{(2)}, \ x \in \mathcal{X}, f \in \mathcal{U}_0 . \tag{7}$$

If $\mathcal{U}_0$ is a $*$-subalgebra and a left coideal of the Hopf $*$-algebra $\mathcal{U}$, then the algebra $\mathcal{X} \rtimes \mathcal{U}_0$ is a $*$-algebra with involution defined by

$$(x \otimes f)^* = (f_{(1)}^* \triangleright x^*) \otimes f_{(2)}^*, \ x \in \mathcal{X}, f \in \mathcal{U}_0 . \tag{8}$$

All $\mathcal{U}$-module algebras $\mathcal{X}$ occuring in this paper are obtained in the following manner: Let $\mathcal{A}$ be a bialgebra and let $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ be a dual pairing of bialgebras $\mathcal{U}$ and $\mathcal{A}$. If $\mathcal{X}$ is a left $\mathcal{A}$-comodule algebra with coaction $\varphi : \mathcal{X} \to \mathcal{A} \otimes \mathcal{X}$, then $\mathcal{X}$ is a right $\mathcal{U}$-module algebra with right action

$$x \triangleleft f = \langle f, x_{(1)} \rangle x_{(2)}, \ x \in \mathcal{X}, f \in \mathcal{U} . \tag{9}$$

Then the cross relation (2) reads

$$xf = f_{(1)} \langle f_{(2)}, x_{(1)} \rangle x_{(2)}, \ x \in \mathcal{X}, f \in \mathcal{U} . \tag{10}$$

If $\mathcal{X}$ is a left comodule $*$-algebra of a Hopf $*$-algebra $\mathcal{A}$ and if $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ is a dual pairing of Hopf $*$-algebras, then $\mathcal{X}$ is also a right $\mathcal{U}$-module $*$-algebra with right action (9) and so Lemma 2.1 applies.

Similarly, any right $\mathcal{A}$-comodule algebra $\mathcal{X}$ is a left $\mathcal{U}$-module algebra with left action

$$f \triangleright x = x_{(1)} \langle f, x_{(2)} \rangle. \tag{11}$$

In this case the cross relation (7) can be written as

$$fx = x_{(1)} \langle f_{(1)}, x_{(2)} \rangle f_{(2)} \ x \in \mathcal{X}, f \in \mathcal{U} . \tag{12}$$

3 Cross product algebras of the Hopf $*$-algebras $\mathcal{U}_q(gl_2)$ and $\mathcal{U}_q(su_2)$

3.1 $\mathcal{O}(M_q(2))$, $\mathcal{O}(SU_q(2))$, $\mathcal{U}_q(gl_2)$ and $\mathcal{U}_q(su_2)$

The algebra $\mathcal{O}(M_q(2))$ has four generators a, b, c, d with defining relations

$$ab = qba, ac = qca, bd = qdb, cd = qdc, bc = cb, ad - da = \lambda bc . \tag{13}$$

The element $\mathcal{D}_q := ad - qbc \equiv da - q^{-1}bc$ is the quantum determinant. It is well known that $\mathcal{O}(M_q(2))$ is a bialgebra and that its quotient algebra $\mathcal{O}(SL_q(2))$ by the two-sided ideal generated by $\mathcal{D}_q - 1$ is a Hopf algebra.

Let $\mathcal{U}_q(gl_2)$ be the algebra with generators $E, F, K, L, K^{-1}, L^{-1}$ and defining relations

$$KL = LK, \ KK^{-1} = K^{-1}K = K, \ LL^{-1} = L^{-1}L = 1 \ ,$$

$$KEK^{-1} = qE, \ KFK^{-1} = q^{-1}F, \ LE = EL, \ LF = FL \ ,$$

$$EF - FE = \lambda^{-1}(K^2 - K^{-2}).$$

The algebra $\mathcal{U}_q(gl_2)$ is a Hopf algebra with structure maps given by

$$\Delta(E) = E \otimes K + K^{-1} \otimes E, \Delta(F) = F \otimes K + K^{-1} \otimes F \ ,$$
$$\Delta(K) = K \otimes K, \Delta(L) = L \otimes L, \varepsilon(K) = \varepsilon(L) = 1, \varepsilon(E) = \varepsilon(F) = 0,$$
$$S(E) = -qE, S(F) = -q^{-1}F, S(K) = K^{-1}, S(L) = L^{-1} \ .$$

The Hopf algebra $\mathcal{U}_q(sl_2)$ is the subalgebra of $\mathcal{U}_q(gl_2)$ generated by E, F, K and K^{-1}.

There exists a dual pairing $\langle \cdot, \cdot \rangle$ of the Hopf algebra $\mathcal{U}_q(gl_2)$ and the bialgebra $\mathcal{O}(M_q(2))$. It is determined by the values on the generators K, L, E, F and a, b, c, d, respectively. The non-zero values are

$$\langle K, a \rangle = \langle K^{-1}, d \rangle = q^{-1/2}, \langle K, d \rangle = \langle K^{-1}, a \rangle = q^{1/2}, \langle E, c \rangle = \langle F, b \rangle = 1. \tag{14}$$
$$\langle L, a \rangle = \langle L, d \rangle = p, \ \langle L^{-1}, a \rangle = \langle L^{-1}, d \rangle = p^{-1} \ . \tag{15}$$

Moreover, (14) gives a dual pairing of the Hopf algebras $\mathcal{U}_q(sl_2)$ and $\mathcal{O}(SL_q(2))$.

Suppose in addition that q and p are real. Then $\mathcal{O}(M_q(2))$ is a $*$-bialgebra, $\mathcal{O}(SL_q(2))$ is a Hopf $*$-algebra, denoted by $\mathcal{O}(SU_q(2))$, and $\mathcal{U}_q(gl_2)$ is a Hopf $*$-algebra, denoted again by $\mathcal{U}_q(gl_2)$, with algebra involutions defined by

$$a^* = d, b^* = -qc \text{ and } E^* = F, K^* = K, L^* = L^{-1}. \tag{16}$$

The subalgebra $\mathcal{U}_q(sl_2)$ of $\mathcal{U}_q(gl_2)$ is a Hopf $*$-algebra denoted by $\mathcal{U}_q(su_2)$. Note that the dual pairing (14)–(15) of $\mathcal{U}_q(gl_2)$ and $\mathcal{O}(M_q(2))$ satisfies

$$\overline{\langle S(f)^*, x \rangle} = \langle f, x^* \rangle, \ f \in \mathcal{U}_q(gl_2), \ x \in \mathcal{O}(M_q(2)). \tag{17}$$

3.2 The cross product $*$-algebras $\mathcal{U}_\mathbf{q}(\mathbf{gl_2}) \ltimes \mathcal{O}(\mathbf{M_q}(\mathbf{2}))$ and $\mathcal{O}(\mathbf{M_q}(\mathbf{2})) \rtimes \mathcal{U}_\mathbf{q}(\mathbf{gl_2})$

Since $\mathcal{O}(M_q(2))$ is a left and right comodule algebra with respect to the comultiplication, $\mathcal{O}(M_q(2))$ is a right and left $\mathcal{U}_q(gl_2)$-module algebra with actions given by (9) and (11), respectively. Hence the cross product algebras $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(M_q(2))$ and $\mathcal{O}(M_q(2)) \rtimes \mathcal{U}_q(gl_2)$ are well defined.

By (14), (15) and (10), the generators E, F, K, L and a, b, c, d satisfy the following cross relations in the *right cross product algebra* $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(M_q(2))$:

$$aE = q^{-1/2}Ea, bE = q^{-1/2}Eb \ , \tag{18}$$

$$cE = q^{1/2}Ec + K^{-1}a, dE = q^{1/2}Ed + K^{-1}b \ , \tag{19}$$

$$aF = q^{-1/2}Fa + K^{-1}c, bF = q^{-1/2}Fb + K^{-1}d \ , \tag{20}$$

$$cF = q^{1/2}Fc, dF = q^{1/2}Fd \ , \tag{21}$$

$$aK = q^{-1/2}Ka, bK = q^{-1/2}Kb, cK = q^{1/2}Kc, dK = q^{1/2}Kd \ , \tag{22}$$

$$aL = pLa, bL = pLb, cL = pLc, dL = pLd \ . \tag{23}$$

The cross relations of the *left cross product algebra* $\mathcal{O}(M_q(2)) \rtimes \mathcal{U}_q(gl_2)$ are:

$$Ea = q^{1/2}aE + bK, Eb = q^{-1/2}bE \ ,$$

$$Ec = q^{1/2}cE + dK, Ed = q^{-1/2}Ed \ ,$$

$$Fa = q^{1/2}aF, Fb = q^{-1/2}bF + aK \ ,$$

$$Fc = q^{1/2}cF, Fd = q^{-1/2}dF + cK \ ,$$

$$Ka = q^{-1/2}aK, Kb = q^{1/2}bK, Kc = q^{-1/2}cK, Kd = q^{1/2}dK \ ,$$

$$La = paL, Lb = pbL, Lc = pcL, Ld = pdL \ .$$

The two cross product algebras are isomorphic as shown by Lemma 3.1 below.

Suppose that q and p are real. Since the dual pairing of $\mathcal{U}_q(gl_2)$ and $\mathcal{O}(M_q(2))$ satisfies condition (17), $\mathcal{O}(M_q(2))$ is a right $\mathcal{U}_q(gl_2)$-module $*$-algebra. Hence, by Lemma 2.1 and its left handed counterpart, the cross product algebras $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(M_q(2))$ and $\mathcal{O}(M_q(2)) \rtimes \mathcal{U}_q(gl_2)$ are $*$-algebras.

Some simple algebraic facts about the above algebras are collected in the next lemma. Its proof is straightforward and will be omitted.

Lemma 3.1 (i) *The quantum determinant* $\mathcal{D}_q = ad - qbc$ *is a central element of the algebras* $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(M_q(2))$ *and* $\mathcal{O}(M_q(2)) \rtimes \mathcal{U}_q(su_2)$.
(ii) *There is an algebra isomorphism* θ *of* $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(M_q(2))$ *onto* $\mathcal{O}(M_q(2)) \rtimes \mathcal{U}_q(gl_2)$ *such that*

$$\theta(a) = a, \theta(d) = d, \theta(b) = -qc, \theta(c) = -q^{-1}b , \tag{24}$$

$$\theta(E) = F, \theta(F) = E, \theta(K) = K^{-1}, \theta(L) = L^{-1} . \tag{25}$$

(iii) *If q and p are real, then θ is a $*$-isomorphism of the $*$-algebras* $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(M_q(2))$ *and* $\mathcal{O}(M_q(2)) \rtimes \mathcal{U}_q(gl_2)$.

Another remarkable property of the isomorphism θ is that its inverse θ^{-1} is given by the same formulas as θ.

3.3 The cross product $*$-algebra $\mathcal{U}_{\mathbf{q}}(\mathbf{gl_2}) \ltimes \hat{\mathcal{O}}(\mathbb{C}_{\mathbf{q}}^2)$

In the rest of this section we suppose that $q, p \in \mathbb{R}$, $p \neq 0$, and $0 < q < 1$.

Let us rename the $*$-algebra $\mathcal{O}(M_q(2))$ defined above by $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ and set $z_1 := b$ and $z_2 := d$. By restating the definition of $\mathcal{O}(M_q(2))$ we see that $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ is the $*$-algebra with four generators z_1, z_2, z_1^*, z_2^* and defining relations

$$z_1 z_2 = q z_2 z_1, \ z_1 z_2^* = q^{-1} z_2^* z_1, z_1^* z_2^* = q^{-1} z_2^* z_1^*, \ z_2 z_1^* = q^{-1} z_1^* z_2 ,$$
$$z_1^* z_1 = z_1 z_1^*, \ z_2^* z_2 - z_2 z_2^* = (q^{-2} - 1) z_1^* z_1 .$$

The first equation $z_1 z_2 = q z_2 z_1$ is just the defining relation of the coordinate algebra $\mathcal{O}(\mathbb{C}_q^2)$ of the quantum plane. The $*$-algebra $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ introduced above is the left handed version of the realification $\mathcal{O}(\mathbb{C}_q^2)_{Re}$ of the algebra $\mathcal{O}(\mathbb{C}_q^2)$ as defined in [KS], p. 391; see also Proposition 9.1.5 therein.

Let $\mathbb{C}[\mathcal{R}_q]$ be the $*$-algebra of polynomials in a hermitian generator $\mathcal{R}_q$. It is easy to check that there exists a unique injective $*$-homomorphism $\psi : \hat{\mathcal{O}}(\mathbb{C}_q^2) \to \mathcal{O}(SU_q(2)) \otimes \mathbb{C}[\mathcal{R}_q]$ such that

$$\psi(z_1) = b\mathcal{R}_q, \ \psi(z_2) = d\mathcal{R}_q . \tag{26}$$

Clearly, ψ is not surjective, because $\mathcal{R}_q$ is not in the image of ψ. We shall consider $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ as a $*$-subalgebra of $\mathcal{O}(SU_q(2)) \otimes \mathbb{C}[\mathcal{R}_q]$ by identifying $x \in \hat{\mathcal{O}}(\mathbb{C}_q^2)$ with $\psi(x)$. Then we have

$$\mathcal{R}_q^2 = z_1^* z_1 + z_2^* z_2. \tag{27}$$

Thus, $\mathcal{O}(SU_q(2))$ is just the coordinate algebra of the quantum unit sphere, which is obtained by adding to the relations of $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ the equation $\mathcal{R}_q^2 = 1$.

In terms of the generators z_1, z_2, z_1^*, z_2^* the cross commutation relations of the *right cross product $*$-algebra* $\mathcal{U}_q(gl_2) \ltimes \hat{\mathcal{O}}(\mathbb{C}_q^2)$ are as follows:

$$z_1 E = q^{-1/2} E z_1, \, z_1^* E = q^{1/2} E z_1^* - q K^{-1} z_2^* , \tag{28}$$

$$z_2 E = q^{1/2} E z_2 + K^{-1} z_1, \, z_2^* E = q^{-1/2} E z_2^* , \tag{29}$$

$$z_1 F = q^{-1/2} F z_1 + K^{-1} z_2, \, z_1^* F = q^{1/2} F z_1^* , \tag{30}$$

$$z_2 F = q^{1/2} F z_2, \, z_2^* F = q^{-1/2} F z_2^* - q^{-1} K^{-1} z_1^* , \tag{31}$$

$$z_1 K = q^{-1/2} K z_1, \, z_1^* K = q^{1/2} K z_1^*, \, z_2 K = q^{1/2} K z_2, \, z_2^* K = q^{-1/2} K z_2^* , \tag{32}$$

$$z_1 L = p L z_1, \, z_1^* L = p L z_1^*, \, z_2 L = p L z_2, \, z_2^* L = p L z_2^* . \tag{33}$$

For the element $\mathcal{R}_q^2 \in \hat{\mathcal{O}}(\mathbb{C}_q^2)$ we have $\mathcal{R}_q^2 K = K \mathcal{R}_q^2$ and $\mathcal{R}_q^2 L = p^2 L \mathcal{R}_q^2$.

3.4   Two cross product $*$-algebras containing $\mathcal{O}(\mathbf{SU_q(2)})$

Recall that the Hopf $*$-algebra $\mathcal{O}(SU_q(2))$ is the Hopf algebra $\mathcal{O}(SL_q(2))$ with the involution given by $a^* = d$ and $b^* = -qc$. Hence $\mathcal{O}(SU_q(2))$ is a right $\mathcal{U}_q(su_2)$-module $*$-algebra with right action (9). The corresponding *right cross product $*$-algebra* $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ has the cross relations (18)–(22).

We also study another cross product $*$-algebra of $\mathcal{O}(SU_q(2))$ where $\mathcal{U}_q(su_2)$ is replaced by a smaller $*$-subalgebra. The quantum tangent space of the $3D$-calculus (see [W] or [KS], p. 407) on $SU_q(2)$ is spanned by the elements

$$X_0 := q^{-1/2} FK, X_2 := q^{1/2} EK, X_1 := (1 - q^{-2})^{-1}(1 - K^4)$$

of $\mathcal{U}_q(sl_2)$. These elements satisfy the relations $X_0^* = X_2$, $X_0^* = X_1$ and

$$q^2 X_1 X_0 - q^{-2} X_0 X_1 = (1 + q^2) X_0 , \tag{34}$$

$$q^2 X_2 X_1 - q^{-2} X_1 X_2 = (1 + q^2) X_2 , \tag{35}$$

$$q X_2 X_0 - q^{-1} X_0 X_2 = -q^{-1} X_1 . \tag{36}$$

Let $\mathcal{U}_0$ be the unital subalgebra of $\mathcal{U}_q(sl_2)$ generated by the elements X_0, X_2, X_1. Since the $3D$-calculus is a $*$-calculus on $SU_q(2)$, $\mathcal{U}_0$ is a $*$-invariant right coideal of $\mathcal{U}_q(su_2)$, so the $*$-algebra $\mathcal{U}_0 \ltimes \mathcal{O}(SU_q(2))$ is well defined. The cross relations of the generators $X_0, X_2, Y_1 := 1 - (1 - q^{-2}) X_1$ and a, b, c, d of the

right cross product $*$-algebra $\mathcal{U}_0 \ltimes \mathcal{O}(SU_q(2))$ read as follows:

$$aX_2 = q^{-1}X_2a, bX_2 = q^{-1}X_2b, \tag{37}$$

$$cX_2 = qX_2c + a, dX_2 = qX_2d + b, \tag{38}$$

$$aX_0 = q^{-1}X_0a + c, bX_0 = q^{-1}X_0b + d, \tag{39}$$

$$cX_0 = qX_0c, dX_0 = qX_0d, \tag{40}$$

$$aY_1 = q^{-2}Y_1a, bY_1 = q^{-2}Y_1b, cY_1 = q^2Y_1c, dY_1 = q^2Y_1d. \tag{41}$$

3.5   The cross product $*$-algebra $\mathcal{U}_{\mathbf{q}}(\mathbf{gl_2}) \ltimes \mathcal{O}(\mathbb{R}^3_{\mathbf{q}})$

Let $\mathcal{O}(\mathbb{R}^3_q)$ be the $*$-algebra with generators x_1, x_2, x_3, defining relations

$$x_1x_2 = q^2x_2x_1, \ \ x_2x_3 = q^2x_3x_2, \ \ x_3x_1 - x_1x_3 = \lambda x_2^2, \tag{42}$$

and involution $x_1^* = q^{-1}x_3, x_2^* = x_2$, see [FRT] or [KS], Proposition 9.14(ii).

   This $*$-algebra is a left and right $\mathcal{O}(M_q(2))$-comodule $*$-algebra with left coaction $\phi_L(x_i) = \sum_j v_{ij}(-q)^{i-j} \otimes x_j$ and right coaction $\phi_R(x_i) = \sum_j x_j \otimes v_{ji}$, where the matrix $v = (v_{ij})$ is

$$v = \begin{pmatrix} a^2 & q'ab & -b^2 \\ q'ac & da + qbc & -q'bd \\ -c^2 & -q'cd & d^2 \end{pmatrix}$$

and $q' = (1 + q^{-2})^{1/2}$. Note that v is just the matrix of the spin 1 corepresentation of $\mathcal{O}(SU_q(2))$ when the elements a, b, c, d are taken as generators of $\mathcal{O}(SU_q(2))$. Thus, $\mathcal{O}(\mathbb{R}^3_q)$ is a right and left $\mathcal{U}_q(gl_2)$-module $*$-algebra with actions given by (9) and (11).

   The *right cross product $*$-algebra* $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(\mathbb{R}^3_q)$ has the cross relations:

$$x_1E = q^{-1}Ex_1, x_2E = Ex_2 - q\gamma K^{-1}x_1, x_3E = qEx_3 + q\gamma K^{-1}x_2, \tag{43}$$

$$x_1F = q^{-1}Fx_1 - q^{-1}\gamma K^{-1}x_2, x_2F = Fx_2 + q^{-1}\gamma K^{-1}x_3, x_3F = qFx_3, \tag{44}$$

$$x_1K = q^{-1}Kx_1, \ x_2K = Kx_2, \ x_3K = qKx_3, \tag{45}$$

$$x_1L = p^2Lx_1, \ x_2L = p^2Lx_2, \ x_3L = p^2Lx_3. \tag{46}$$

   Some properties of the algebra $\mathcal{O}(\mathbb{R}^3_q)$ are collected in the following lemma. We omit the proof.

Lemma 3.2 (i) $\mathcal{Q}^2_q := q^{-1}x_1x_3 + qx_3x_1 + x_2^2 = (1 + q^{-2})x_3^*x_3 + q^2x_2^2$ *belongs to the center of the algebra $\mathcal{O}(\mathbb{R}^3_q)$ and $\phi_L(\mathcal{Q}^2_q) = \mathcal{D}^2_q \otimes \mathcal{Q}^2_q$.*

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(ii) *There is an injective $*$-homomorphism ρ of $\mathcal{O}(\mathbb{R}_q^3)$ into $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ such that*
$\rho(x_1)=(1+q^2)^{-1/2}(z_2^{*2}-q^{-1}z_1^2)$, $\rho(x_2)=z_1^*z_2^*+z_2z_1$, $\rho(x_3)=(1+q^2)^{-1/2}(qz_2^2-z_1^{*2})$.
(iii) *$(id\otimes\rho)\circ\phi_L=\Delta\circ\rho$, where Δ is the comultiplication of $\mathcal{O}(M_q(2))=\hat{\mathcal{O}}(\mathbb{C}_q^2)$.*
(iv) *There is a $*$-isomorphism ϑ of $\mathcal{U}_q(gl_2)\ltimes\mathcal{O}(\mathbb{R}_q^3)$ onto $\mathcal{O}(\mathbb{R}_q^3)\rtimes\mathcal{U}_q(gl_2)$ such that $\vartheta(x_i)=x_i$, $\vartheta(E)=F$, $\vartheta(F)=E$, $\vartheta(K)=K^{-1}$ and $\vartheta(L)=L^{-1}$.*

From Lemma 3.2,(ii) and (iii), it follows that $\mathcal{O}(\mathbb{R}_q^3)$ and $\mathcal{U}_q(gl_2)\ltimes\mathcal{O}(\mathbb{R}_q^3)$ are $*$-subalgebras of $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ and $\mathcal{U}_q(gl(2)\ltimes\hat{\mathcal{O}}(\mathbb{C}_q^2)$, respectively, if we identify $x\in\mathcal{O}(\mathbb{R}_q^3)$ and $\rho(x)\in\hat{\mathcal{O}}(\mathbb{C}_q^2)$.

The quotient $*$-algebra $\mathcal{O}(S_q^2)$ of $\mathcal{O}(\mathbb{R}_q^3)$ by the ideal generated by the central hermitian element $\mathcal{Q}_q^2-1$ is called the coordinate algebra of the quantum unit sphere S_q^2 of $\mathbb{R}_q^3$. Let y_i denote the image of the generator x_i of $\mathcal{O}(\mathbb{R}_q^3)$ under the quotient map. The defining relations of the algebra $\mathcal{O}(S_q^2)$ are

$$y_1y_2 = q^2y_2y_1, \; y_2y_3 = q^2y_3y_2, \; y_3y_1 - y_1y_3 = \lambda y_2^2,$$
$$q^{-1}y_1y_3 + qy_3y_1 + y_2^2 = 1 \; . \tag{47}$$

The quantum sphere S_q^2 of $\mathbb{R}_q^3$ is one of Podles' quantum spheres S_{qc}^2. More precisely, it is the quantum sphere $S_{q\infty}^2$ in [Po] and S_{q0}^2 in [KS], Section 4.5.

Since $\phi_L(\mathcal{Q}_q^2) = \mathcal{D}_q^2\otimes\mathcal{Q}_q^2$ by Lemma 3.2(i), the left coaction ϕ_L of $\mathcal{O}(M_q(2))$ on $\mathcal{O}(\mathbb{R}_q^3)$ passes to a left coaction of the quotient algebras $\mathcal{O}(SU_q(2))$ on $\mathcal{O}(S_q^2)$. Thus, $\mathcal{O}(S_q^2)$ is a left $\mathcal{O}(SU_q(2))$-comodule and hence right $\mathcal{U}_q(su_2)$-module $*$-algebra. The corresponding *right cross product $*$-algebra* $\mathcal{U}_q(su_2)\ltimes\mathcal{O}(S_q^2)$ has the cross relations (43)–(45) with x_i replaced by y_i.

The assertions (ii)–(iv) of Lemma 3.2 have their counterparts for the quantum sphere S_q^2. The map ϑ from Lemma 3.2(iv) passes to a $*$-isomorphism of the right and left cross product $*$-algebras $\mathcal{U}_q(su_2)\ltimes\mathcal{O}(S_q^2)$ and $\mathcal{O}(S_q^2)\rtimes\mathcal{U}_q(su_2)$. There exists an injective $*$-isomorphism ρ from $\mathcal{O}(S_q^2)$ into $\mathcal{O}(SU_q(2))$ such that $(id\otimes\rho)\circ\phi_L=\Delta\circ\rho$ and

$$\rho(y_1) = (1+q^2)^{-1/2}(a^2-q^{-1}b^2), \rho(y_2) = db-qca, \rho(y_3) = (1+q^2)^{-1/2}(qd^2-q^2c^2).$$

Thus, if we identify $x\in\mathcal{O}(S_q^2)$ with $\rho(x)\in\mathcal{O}(SU_q(2))$, then $\mathcal{U}_q(su_2)\ltimes\mathcal{O}(S_q^2)$ becomes a $*$-subalgebra of $\mathcal{U}_q(su_2)\ltimes\mathcal{O}(SU_q(2))$.

Let $\mathbb{C}[\mathcal{Q}_q]$ be the $*$-algebra of polynomials in $\mathcal{Q}_q = \mathcal{Q}_q^*$. There is an injective $*$-homomorphism $\psi : \mathcal{O}(\mathbb{R}_q^3) \to \mathcal{O}(S_q^2)\otimes\mathbb{C}[\mathcal{Q}_q]$ such that $\psi(x_i) = y_i\mathcal{Q}_q$, $i = 1,2,3$. By identifying $x\in\mathcal{O}(\mathbb{R}_q^3)$ with $\psi(x)$, $\mathcal{O}(\mathbb{R}_q^3)$ becomes a $*$-subalgebra of $\mathcal{O}(S_q^2)\otimes\mathbb{C}[\mathcal{Q}_q]$.

4 Preliminaries on representations

4.1 Three auxiliary lemmas

In this subsection q is a positive real number such that $q \neq 1$.

Lemma 4.1 *Up to unitary equivalence each isometry w on a Hilbert space $\mathcal{H}$ is of the following form: There exist Hilbert subspaces $\mathcal{H}^u$ and $\mathcal{H}^s_0$ of $\mathcal{H}$ and a unitary operator w_u on $\mathcal{H}^u$ such that $w = w_u \oplus w_s$ on $\mathcal{H} = \mathcal{H}^u \oplus \mathcal{H}^s$, where $\mathcal{H}^s = \oplus_{n=0}^{\infty} \mathcal{H}^s_n$, $\mathcal{H}^s_n = \mathcal{H}^s_0$, and $w_s \eta_n = \eta_{n+1}$, $\eta_n \in \mathcal{H}^s_n$. This decomposition of w is unique, because $\mathcal{H}^u = \cap_{n=0}^{\infty} w^n \mathcal{H}$ and $\mathcal{H}^s$ is the closed linear span of $w^n(\ker w^*), n \in \mathbb{N}_0$.*

Lemma 4.1 is proved in [SF], Theorem 1.1. The decomposition $w = w_u \oplus w_s$ is called the *Wold decomposition* of w. The operator w_u is the unitary part of w and w_s is a unilateral shift operator of multiplicity $\dim(\mathcal{H}^s_0)$.

Lemma 4.2 *Let A be a self-adjoint operator and let w be an isometry on a Hilbert space $\mathcal{H}$ such that*

$$qwA \subseteq Aw. \tag{48}$$

Then the Wold decomposition $w = w_u \oplus w_s$ on $\mathcal{H} = \mathcal{H}^u \oplus \mathcal{H}^s$ reduces the operator A, that is, there are self-adjoint operators A^u on $\mathcal{H}^u$ and A^s on $\mathcal{H}^s$ such that $A = A^u \oplus A^s$, and we have:
(i) If A is strictly positive, then there exists a self-adjoint operator A^u_0 on a Hilbert space $\mathcal{H}^u_0$ with $\sigma(A^u_0) \sqsubseteq (q, 1]$ if $q < 1$ and $\sigma(A^u_0) \sqsubseteq (q^{-1}, 1]$ if $q > 1$ such that, up to unitary equivalence, $\mathcal{H}^u = \oplus_{n=-\infty}^{\infty} \mathcal{H}^u_n$, where $\mathcal{H}^u_n := \mathcal{H}^u_0$,

$$A^u \eta_n = q^n A^u_0 \eta_n, \ w_u \eta_n = \eta_{n+1}, \quad \text{for } \eta_n \in \mathcal{H}^u_n, n \in \mathbb{Z} \ .$$

(ii) There is a self-adjoint operator A^s_0 on the Hilbert space $\mathcal{H}^s_0$ such that

$$A^s \eta_n = q^n A^s_0 \eta_n, \ w_s \eta_n = \eta_{n+1}, \quad \text{for } \eta_n \in \mathcal{H}^s_n, n \in \mathbb{N}_0 \ .$$

Proof. From the functional calculus of self-adjoint operators it follows that equation (48) implies that $w\varphi(qA) = \varphi(A)w$ for $\varphi \in L^{\infty}(\mathbb{R})$. Therefore, for any $\varphi \in L^{\infty}(\mathbb{R})$, the subspace $\mathcal{H}^u := \cap_{n=0}^{\infty} w^n \mathcal{H}$ is $\varphi(A)$-invariant, and therefore so is its orthogonal complement $\mathcal{H}^s$. This implies that A decomposes as $A = A^u \oplus A^s$ with respect to the orthogonal sum $\mathcal{H} = \mathcal{H}^u \oplus \mathcal{H}^s$.

Let $e(\mu)$ denote the spectral projections of A^u. Let $\mathcal{H}^u_n := e((q^{n+1}, q^n])\mathcal{H}^u$ and $A^u_n := A^u \upharpoonright \mathcal{H}^u_n$, $n \in \mathbb{Z}$. Since A^u is strictly positive, $\mathcal{H}^u = \oplus_{n=-\infty}^{\infty} \mathcal{H}^u_n$. Since w_u is unitary, (48) implies that $A^u = qw_u A^u w^*_u$ and hence $\varphi(A^u) =$

$w_u \varphi(q A^u) w_u^*$ for $\varphi \in L^\infty(\mathbb{R})$. This yields $w_u \mathcal{H}_n^u = \mathcal{H}_{n+1}^u$. Thus, up to unitary equivalence, we can assume that $\mathcal{H}_n^u = \mathcal{H}_0^u$ and $w_u \eta_n = \eta_{n+1}$ for $\eta_n \in \mathcal{H}_n^u$. Thus, $A^u \eta_n = q^n w_u^n A^u w_u^{*n} \eta_n = q^n w_u^n A_0^u \eta_0 = q^n A_0^u \eta_n$ which proves (i).

Since $w_s \varphi(q A^s) = \varphi(A^s) w_s$ for $\varphi \in L^\infty(\mathbb{R})$ by (48), $\varphi(A^s)$ leaves $\ker w_s^{n*} = \mathcal{H}_0^s + \cdots + \mathcal{H}_{n-1}^s$ invariant. Since this is true for all $\varphi \in L^\infty(\mathbb{R})$, it follows that $\varphi(A^s)$ leaves each space $\mathcal{H}_m^s$ invariant. Setting $A_0^s = A^s \lceil \mathcal{H}_0^s$, relation (48) gives $A^s \eta_n = q^n A_0^s \eta_n$. This proves (ii). $\qquad \square$

Lemma 4.3 *Let x be a closed operator on Hilbert space $\mathcal{H}$. Then we have $\mathcal{D}(xx^*) = \mathcal{D}(x^*x)$, this domain is dense in $\mathcal{H}$, and the relation*

$$xx^* - q^2 x^* x = 1 \tag{49}$$

holds if and only if x is unitarily equivalent to an orthogonal direct sum of operators of the following forms:

(I) *for any $q > 0$: $x\eta_n = ((1 - q^{2n})/(1 - q^2))^{1/2} \eta_{n-1}$*
on $\mathcal{H} = \oplus_{n=0}^\infty \mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}_0$.

$(II)_A$ *for $0 < q < 1$: $x\eta_n = (1 - q^2)^{-1/2} \alpha_n(A) \eta_{n-1}$*
on $\mathcal{H} = \oplus_{n=-\infty}^\infty \mathcal{H}_n, \mathcal{H}_n = \mathcal{H}_0$, where A is a self-adjoint operator on the
Hilbert space $\mathcal{H}_0$ such that $\sigma(A) \sqsubseteq (q^2, 1]$ and $\alpha_n(A) := (1 + q^{2n} A)^{1/2}$.

$(III)_u$ *for $0 < q < 1$: $x = (1 - q^2)^{-1/2} u$, where u is a unitary operator on $\mathcal{H}$.*

Proof. Clearly, the operators x have the stated properties. That all *irreducible* operators x are of one of the above form was proved in [CGP]. The general case follows by decomposition theory or by modifying the proof in [CGP]. $\qquad \square$

Representation (I) is usually called the *Fock representation*. Note that for $q > 1$ the Fock representation is the only representation of relation (49).

4.2 Representations of the coordinate ∗-algebras

Suppose that $0 < q < 1$. Let $\mathcal{H}_0$ and $\mathcal{G}$ be Hilbert spaces and set $\mathcal{H} = \oplus_{n=0}^\infty \mathcal{H}_n$, where $\mathcal{H}_n := \mathcal{H}_0$. One checks that the following formulas define ∗-representations of the corresponding ∗-algebras on the Hilbert space $\mathcal{G} \oplus \mathcal{H}$:

$\mathcal{O}(SU_q(2)):\ a = v,\ d = v^*,\ b = c = 0\ $ on $\mathcal{G}$,

$$a\eta_n = \lambda_n \eta_{n-1},\ d\eta_n = \lambda_{n+1} \eta_{n+1},\ b\eta_n = q^{n+1} w \eta_n,\ c\eta_n = -q^n w^* \eta_n, \tag{50}$$

where w and v are unitary operators on $\mathcal{H}_0$ and $\mathcal{G}$, respectively.

$\hat{\mathcal{O}}(\mathbb{C}_q^2):$ $z_1 = z_1^* = 0,$ $z_2 = M,$ $z_2^* = M^*$ on $\mathcal{G}$,

$$z_1\eta_n = q^{n+1}wA\eta_n, z_1^*\eta_n = q^{n+1}w^*A\eta_n, z_2\eta_n = \lambda_{n+1}A\eta_{n+1}, z_2^*\eta_n = \lambda_n A\eta_{n-1},$$
(51)

where M is a normal operator on $\mathcal{G}$, A is a strictly positive self-adjoint operator and w is a unitary on $\mathcal{H}_0$ such that $wAw^* = A$.

$\mathcal{O}(S_q^2):$ $y_1 = (1+q^2)^{-1/2}u^*,$ $y_2 = 0,$ $y_3 = (1+q^{-2})^{-1/2}u$ on $\mathcal{G}$,

$$y_1\eta_n = (1+q^2)^{-1/2}\lambda_{2n}\eta_{n-1}, y_2\eta_n = q^{2n+1}w\eta_n \ ,$$
(52)

$$y_3\eta_n = q(1+q^2)^{-1/2}\lambda_{2(n+1)}\eta_{n+1} \ ,$$
(53)

where u is a unitary operator on $\mathcal{G}$ and w is a self-adjoint unitary on $\mathcal{H}_0$.

$\mathcal{O}(\mathbb{R}_q^3):$ $x_1 = q^{-1}M^*,$ $x_2 = 0,$ $x_3 = M$ on $\mathcal{G}$,

$$x_1\eta_n = (1+q^2)^{-1/2}\lambda_{2n}A\eta_{n-1}, x_2\eta_n = q^{2n+1}wA\eta_n \ ,$$
(54)

$$x_3\eta_n = q(1+q^2)^{-1/2}\lambda_{2(n+1)}A\eta_{n+1} \ ,$$
(55)

where M is a normal operator on $\mathcal{G}$ and A is a strictly positive self-adjoint operator and w is a self-adjoint unitary on $\mathcal{H}_0$ such that $wAw^* = A$.

Lemma 4.4 *Any $*$-representation of $\mathcal{O}(SU_q(2))$ or $\mathcal{O}(S_q^2)$ is up to unitary equivalence of the above form.*

Proof. For $\mathcal{O}(SU_q(2))$ the assertion is Proposition 4.19 in [KS]. We sketch the proof for $\mathcal{O}(S_q^2)$. The third defining relation of $\mathcal{O}(S_q^2)$ yields

$$y_3^*y_3 - y_3y_3^* = (1-q^2)y_2^2 \ .$$
(56)

Combining the latter with (47), we obtain

$$y_3^*y_3 - q^4y_3y_3^* = q^2(1-q^2) \ .$$
(57)

By (47), y_3 and y_2 are bounded. Hence $\mathcal{G} := \ker y_2$ is reducing and $u := (1+q^{-2})^{1/2}y_3\lceil\mathcal{G}$ is unitary on $\mathcal{G}$ by (56) and (57).

Assume now that $\ker y_2 = \{0\}$. The representations of relation (57) are given by Lemma 4.3. For the series (II) and (III) the operator $y_3^*y_3 - y_3y_3^*$ is not strictly positive which contradicts (56). Thus, only the Fock

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representation (I) is possible, so y_3 has the above form. Let $y_2 = v|y_2|$ be the polar decomposition of y_2. Since y_2 is self-adjoint and ker $y_2 = \{0\}$, v is self-adjoint and unitary. From the formula for y_3 and (56) we obtain $y_2^2 \eta_n = q^{4n+2} \eta_n$ and so $|y_2|\eta_n = q^{2n+1}\eta_n$. Since $v|y_2|v^* = |y_2|$, v leaves each space $\mathcal{H}_n$ invariant. Hence there are self-adjoint unitaries v_n on $\mathcal{H}_n$ such that $v\eta_n = v_n\eta_n$. From the relation $y_2 y_3 = q^2 y_3 y_2$ it follows that $w := v_0 = v_n$ for all n. □

Let us turn now to the $*$-algebras $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ and $\mathcal{O}(\mathbb{R}_q^3)$. Suppose we have a $*$-representation of $\mathcal{O}(SU_q(2))$ resp. $\mathcal{O}(S_q^2)$ and a (possibly unbounded) self-adjoint operator R commuting with all representation operators. Obviously, we then obtain a $*$-representation of $\mathcal{O}(SU_q(2)) \otimes \mathbb{C}[\mathcal{R}_q]$ resp. $\mathcal{O}(S_q^2) \otimes \mathbb{C}[\mathcal{Q}_q]$ and so of its $*$-subalgebra $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ resp. $\mathcal{O}(\mathbb{R}_q^3)$ on the domain $\mathcal{D} := \cap_{n=0}^{\infty} \mathcal{D}(R^n)$ which maps $\mathcal{R}_q$ resp. $\mathcal{Q}_q$ into $R\lceil\mathcal{D}$. We shall think of $\mathcal{R}_q$ and $\mathcal{Q}_q$ as *quantum radii* of the quantum vector spaces $\mathbb{C}_q^2$ and $\mathbb{R}_q^3$, respectively. Thus it is natural to require that these elements are represented by a positive operator R. Let us call $*$-representations of $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ resp. $\mathcal{O}(\mathbb{R}_q^3)$ of this form *admissible*.

It is easy to show that the $*$-representations of $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ and $\mathcal{O}(\mathbb{R}_q^3)$ listed above are admissible and that any admissible $*$-representation of $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ resp. $\mathcal{O}(\mathbb{R}_q^3)$ is up to unitary equivalence of the above form.

4.3 Integrable representations of $\mathcal{U}_\mathbf{q}(\mathbf{su_2})$

For later use let us restate some well known facts (see e.g. [KS]). The irreducible unitary corepresentations of $\mathcal{O}(SU_q(2))$ are labeled by numbers $l \in \frac{1}{2}\mathbb{N}_0$. By (11), each such corepresentation gives an irreducible $*$-representation T_l of $\mathcal{U}_q(su_2)$. The $*$-representation T_l acts on a $(2l+1)$-dimensional Hilbert space V_l with orthonormal basis $\{e_j; j = -l, -l+1, \ldots, l\}$ by the formulas:

$$T_l(K)e_j = q^j e_j, T_l(E)e_j = \alpha_{j+1,l} e_{j+1}, T_l(F)e_j = \alpha_{j,l} e_{j-1}, \qquad (58)$$

where $e_{l+1}=e_{-l-1}=0$ and $\alpha_{jl} := ([l+j]_q[l-j+1]_q)^{1/2}$. For the Casimir element

$$C_q := EF + \lambda^{-2}(q^{-1}K^2 + qK^{-2} - 2) = FE + \lambda^{-2}(qK^2 + q^{-1}K^{-2} - 2) \quad (59)$$

we have

$$T_l(C_q) = [l + 1/2]_q^2 = \lambda^{-2}(q^{l+1/2} - q^{-l-1/2})^2. \qquad (60)$$

In representation theory (see [Wa], Ch. 5 or [S], Ch. 10), a $*$-representation of a universal enveloping algebra is called integrable if it comes from a unitary representation of the corresponding connected simply connected Lie group. This suggest the following definition.

We say that a closed $*$-representation of $\mathcal{U}_q(su_2)$ on a Hilbert space is *integrable* if it is a direct sum of $*$-representations $T_l, l \in \frac{1}{2}\mathbb{N}_0$. It can be shown that a closed $*$-representation π of $\mathcal{U}_q(su_2)$ is integrable if and only if π is a direct sum of finite dimensional $*$-representations and $\sigma(\overline{\pi(K)}) \subseteq [0, +\infty)$.

5 Invariant functionals and Heisenberg representations

In the rest of this paper we assume that $0 < q < 1$, $p > 0$ and $p \neq 1$.

5.1 Invariant functionals on coordinate algebras

Let $\mathcal{X}$ be a right module algebra of a Hopf algebra $\mathcal{U}$ with right action $\triangleleft$. A linear functional h on $\mathcal{X}$ is called $\mathcal{U}$-*invariant* if $h(x \triangleleft f) = \varepsilon(f)h(x)$ for all $x \in \mathcal{X}$ and $f \in \mathcal{U}$.

Suppose $\mathcal{X}$ is a left comodule algebra of a Hopf algebra $\mathcal{A}$. A linear functional h on $\mathcal{X}$ is said to be $\mathcal{A}$-*invariant* if $(\mathrm{id} \otimes h)\varphi(x) = h(x)1$ for $x \in \mathcal{X}$ or equivalently if $h(x_{(2)})x_{(1)} = h(x)1$ for $x \in \mathcal{X}$.

If $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ is a dual pairing of Hopf algebras, then $\mathcal{X}$ is a right $\mathcal{U}$-module algebra with right action (9). In this case the two invariance concepts are not equivalent, but they are related as follows.

Lemma 5.1 *Let h be a linear functional on $\mathcal{X}$.*
(i) If h is $\mathcal{A}$-invariant, then h is also $\mathcal{U}$-invariant.
(ii) Suppose that $\mathcal{U}$ separates the points of $\mathcal{A}$, that is, if $x \in \mathcal{A}$ and $\langle f, x \rangle = 0$ for all $f \in \mathcal{U}$, then $x = 0$. Then h is $\mathcal{A}$-invariant if h is $\mathcal{U}$-invariant.

Proof. (i) is obvious. We verify (ii) and assume that h is $\mathcal{U}$-invariant. By the $\mathcal{U}$-invariance and (9), we have

$$h(x \triangleleft f) = \langle f, h(x_{(2)})x_{(1)} \rangle = \varepsilon(f)h(x) = \langle f, h(x)1 \rangle$$

for all $f \in \mathcal{U}$ and so $h(x_{(2)})x_{(1)} = h(x)1$. $\qquad\qquad\square$

Now we specialize the preceding to the case $\mathcal{U} = \mathcal{U}_q(su_2)$ and $\mathcal{A} = \mathcal{O}(SU_q(2))$ with the dual pairing given by (14). It is well known (see [KS], p. 113 and p. 128) that there is a unique $\mathcal{A}$-invariant linear functional h such that $h(1) = 1$ on each of the right $\mathcal{A}$-comodule algebras $\mathcal{X} = \mathcal{O}(SU_q(2))$ and $\mathcal{X} = \mathcal{O}(S_q^2)$. This functional h is a state on the $*$-algebra $\mathcal{X}$. For $\mathcal{X} = \mathcal{O}(SU_q(2))$, it is called the Haar state of $SU_q(2)$ and explicitly given by

$$h(a^r b^k c^l) = h(d^r b^k c^l) = \delta_{r0} \delta_{kl} (-1)^k [k+1]_q^{-1}, r, k, l \in \mathbb{N}_0. \qquad (61)$$

Since $\mathcal{U}_q(su_2)$ separates the points of $\mathcal{O}(SU_q(2))$ (see [KS], 4.4), Lemma 5.1 applies and by the preceding we have

Lemma 5.2 *There is a unique $\mathcal{U}_q(su_2)$-invariant linear functional h on $\mathcal{X} = \mathcal{O}(SU_q(2))$ resp. $\mathcal{X} = \mathcal{O}(S_q^2)$ satisfying $h(1) = 1$.*

Next we look for $\mathcal{U}_q(gl_2)$-invariant linear functionals on $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ and $\mathcal{O}(\mathbb{R}_q^3)$. We carry out the construction for $\hat{\mathcal{O}}(\mathbb{C}_q^2)$. Replacing $\mathcal{O}(SU_q(2))$ by $\mathcal{O}(S_q^2)$ and $\mathcal{R}_q^2$ by $\mathcal{Q}_q^2$, the case of $\mathcal{O}(\mathbb{R}_q^3)$ is treated completely similarly. Recall that $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ is a $*$-subalgebra of $\mathcal{O}(SU_q(2)) \otimes \mathbb{C}[\mathcal{R}_q]$ via the $*$-isomorphism ψ defined by (26). In order to "integrate over the quantum plane", we need more functions of $\mathcal{R}_q$ than polynomials. Let $\mathcal{F}$ denote the $*$-algebra of all Borel functions on $(0, +\infty)$. For $\varphi \in \mathcal{F}$ we write $\varphi(\mathcal{R}_q)$ instead of $\varphi(t)$ and consider $\mathbb{C}[\mathcal{R}_q]$ as a $*$-subalgebra of $\mathcal{F}$. Then, $\hat{\mathcal{O}}_e(\mathbb{C}_q^2) := \mathcal{O}(SU_q(2)) \otimes \mathcal{F}$ is a right $\mathcal{U}_q(gl_2)$-module $*$-algebra under the actions $(\varphi \triangleleft L)(\mathcal{R}_q) = \varphi(p\mathcal{R}_q)$ and $\varphi \triangleleft f = \varepsilon(f)\varphi$ for all $\varphi \in \mathcal{F}$ and $f \in \mathcal{U}_q(su_2)$. The cross product $*$-algebra $\mathcal{U}_q(gl_2) \ltimes \hat{\mathcal{O}}_e(\mathbb{C}_q^2)$ has the cross relation $\varphi(\mathcal{R}_q)L = L\varphi(p\mathcal{R}_q)$ and contains $\mathcal{U}_q(gl_2) \ltimes \hat{\mathcal{O}}(\mathbb{C}_q^2)$ as $*$-subalgebra.

Let μ_0 be a positive Borel measure supported on the interval $(p, 1]$ if $p < 1$ resp. $(p^{-1}, 1]$ if $p > 1$. Let $C_c(0, +\infty)$ denote the continuous function on $(0, +\infty)$ with compact support. There is a unique positive Borel measure μ on $(0, +\infty)$ such that $\mu(p\mathfrak{M}) = p\mu(\mathfrak{M})$ for any Borel subset $\mathfrak{M}$ of $(0, +\infty)$. Then there is a $\mathcal{U}_q(gl_2)$-invariant linear functional h_{μ_0} on the $*$-subalgebra $\mathcal{O}(SU_q(2)) \otimes C_c(0, +\infty)$ of $\hat{\mathcal{O}}_e(\mathbb{C}_q^2)$ such that

$$h_{\mu_0}(x\varphi(\mathcal{R}_q)) = h(x) \int_0^\infty \varphi(t) d\mu(t), \ x \in \mathcal{O}(SU_q(2)), \varphi \in C_c(0, +\infty).$$

For instance, if μ_0 is a Dirac measure δ_{t_0} , then μ is supported on the points

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$t_0 p^n$, $n \in \mathbb{Z}$, and $h_{\mu_0}(\varphi(\mathcal{R}_q))$ is given by the Jackson integral

$$h_{\mu_0}(\varphi(\mathcal{R}_q)) = \int_0^\infty \varphi(t) d\mu(t) = \sum_{n=0}^{+\infty} \varphi(t_0 p^n) p^n.$$

5.2 Heisenberg representations of cross product algebras

Let $\mathcal{U}_0 \ltimes \mathcal{X}$ be a right cross product algebra as in Section 2. It is well known that there exists a unique homomorphism π of $\mathcal{U}_0 \ltimes \mathcal{X}$ into the algebra $L(\mathcal{X})$ of linear mappings of $\mathcal{X}$ such that $\pi(x)y = xy$ and $\pi(f)y = y \triangleleft S^{-1}(f)$ for $x, y \in \mathcal{X}$ and $f \in \mathcal{U}_0$. (Indeed, one easily checks that $\pi(x)\pi(f) = \pi(f_{(1)})\pi(x \triangleleft f_{(2)})$, so there is a well defined (!) homomorphism π with these properties.) In this subsection we develop a Hilbert space version of this algebraic fact.

Suppose $\mathcal{X}$ is a $*$-algebra with unit element. Let h be a state on $\mathcal{X}$, i.e. h is a linear functional on $\mathcal{X}$ such that $h(x^*x) \geq 0$ for $x \in \mathcal{X}$ and $h(1) = 1$, and let π_h denote the GNS-representation π_h of $\mathcal{X}$ (see e.g. [S], Section 8.6). By the definition of the GNS-representation, there is a vector v_h in the domain of π_h such that $\mathcal{D}_h = \pi_h(\mathcal{X})v_h$ is dense in the underlying Hilbert space $\mathcal{H}_h$, $\pi_h(x)(\pi_h(y)v_h) = \pi_h(xy)v_h$ for $x, y \in \mathcal{X}$ and

$$h(x) = \langle \pi_h(x)v_h, v_h \rangle, \ x \in \mathcal{X}. \tag{62}$$

Proposition 5.3 *Let $\mathcal{X}$ be a right $\mathcal{U}$-module $*$-algebra of a Hopf $*$-algebra $\mathcal{U}$ and let $\mathcal{U}_0$ be a unital $*$-subalgebra and a right coideal of $\mathcal{U}$. Suppose that h is a $\mathcal{U}_0$-invariant state on $\mathcal{X}$. Then there exists a unique $*$-representation $\tilde{\pi}_h$ of the $*$-algebra $\mathcal{U}_0 \ltimes \mathcal{X}$ on the domain $\mathcal{D}_h = \pi_h(\mathcal{X})v_h$ such that $\tilde{\pi}_h \lceil \mathcal{X}$ is the GNS-representation of $\mathcal{X}$ and $\tilde{\pi}_h(f)v_h = \varepsilon(f)v_h$ for $f \in \mathcal{U}_0$. For $f \in \mathcal{U}_0$ and $x \in \mathcal{X}$, we have*

$$\tilde{\pi}_h(f)(\pi_h(x)v_h) = \pi_h(x \triangleleft S^{-1}(f))v_h. \tag{63}$$

Proof. First we show that the linear mapping $\tilde{\pi}_h(f)$ given by (63) is well defined. Let $x \in \mathcal{X}$ be such that $\pi_h(x)v_h = 0$ and let $f, g \in \mathcal{U}_0$. Using (62),

(5), and the $\mathcal{U}_0$-invariance of h we obtain

$$
\begin{aligned}
\langle \pi_h(x \triangleleft S^{-1}(f))v_h, \pi_h(x \triangleleft S^{-1}(g))v_h \rangle &= h((x \triangleleft S^{-1}(g))^*(x \triangleleft S^{-1}(f))) \\
&= h((x^* \triangleleft g^*)(x \triangleleft S^{-1}(f_{(2)})))\varepsilon(f_{(1)}) = h((x^* \triangleleft g^*)(x \triangleleft S^{-1}(f_{(2)})) \triangleleft f_{(1)}) \\
&= h((x^* \triangleleft g^* f_{(1)})(x \triangleleft S^{-1}(f_{(3)})f_{(2)})) = h((x^* \triangleleft g^* f)x) \\
&= \langle \pi_h(x)v_h, \pi_h((x^* \triangleleft g^* f)^*)v_h \rangle = 0.
\end{aligned}
$$

Setting $f = g$, we get $\pi_h(x \triangleleft S^{-1}(f))v_h = 0$. Hence the linear mapping $\tilde{\pi}_h(f)$ is well defined by (63).

Obviously, the map $f \to \tilde{\pi}_h(f)$ is an algebra homomorphism of $\mathcal{U}_0$ into $L(\mathcal{D}_h)$. From (63) and the definition of the GNS-representation π_h of $\mathcal{X}$ it follows easily that $\tilde{\pi}_h$ extends to an algebra homomorphism, denoted again by $\tilde{\pi}_h$, of $\mathcal{U}_0 \ltimes \mathcal{X}$ into $L(\mathcal{D}_h)$ such that $\tilde{\pi}_h \lceil \mathcal{X} = \pi_h$. Next we show that $\tilde{\pi}_h$ preserves the involution. Let $f \in \mathcal{U}_0$ and $x, y \in \mathcal{X}$. Using (62), (63), (5), and the $\mathcal{U}_0$-invariance of h we compute

$$
\begin{aligned}
\langle \tilde{\pi}_h(f)\pi_h(x)v_h, \pi_h(y)v_h \rangle &= \langle \pi_h(y^*(x \triangleleft S^{-1}(f)))v_h, v_h \rangle = h(y^*(x \triangleleft S^{-1}(f))) \\
&= h(y^*(x \triangleleft S^{-1}(f_{(2)})))\varepsilon(f_{(1)}) = h((y^*(x \triangleleft S^{-1}(f_{(2)})) \triangleleft f_{(1)}) \\
&= h((y^* \triangleleft f_{(1)})(x \triangleleft S^{-1}(f_{(3)})f_{(2)})) = h((y^* \triangleleft f)x) = h((y \triangleleft S^{-1}(f^*))^* x) \\
&= \langle \pi_h((y \triangleleft S^{-1}(f^*))^* x)v_h, v_h \rangle = \langle \pi_h(x)v_h, \tilde{\pi}_h(f^*)\pi_h(y)v_h \rangle.
\end{aligned}
$$

This shows that $\tilde{\pi}_h \lceil \mathcal{U}_0$ is a $*$-representation of the $*$-algebra $\mathcal{U}_0$. The restriction of $\tilde{\pi}_h$ to $\mathcal{X}$ is the GNS-representation and so a $*$-representation. Since the $*$-algebra $\mathcal{U}_0 \ltimes \mathcal{X}$ is generated by $\mathcal{U}_0$ and $\mathcal{X}$ (by Lemma 2.1), $\tilde{\pi}_h$ is a $*$-representation of the $*$-algebra $\mathcal{U}_0 \ltimes \mathcal{X}$ on the domain $\mathcal{D}_h$.

Finally, we prove the uniqueness assertion. Let π be a representation of $\mathcal{U}_0 \ltimes \mathcal{X}$ such that $\pi \lceil \mathcal{X} = \pi_h$ and $\pi(f)v_h = \varepsilon(f)v_h$. Using (3) and the fact that π is an algebra homomorphism we obtain

$$
\begin{aligned}
\pi(f)\pi_h(x)v_h = \pi(fx)v_h &= \pi((x \triangleleft S^{-1}(f_{(2)})f_{(1)})v_h \\
&= \pi(x \triangleleft S^{-1}(f_{(2)}))\pi(f_{(1)})v_h = \pi(x \triangleleft S^{-1}(f))v_h = \tilde{\pi}_h(f)\pi_h(x)v_h
\end{aligned}
$$

for $f \in \mathcal{U}_0$ and $x \in \mathcal{X}$. That is, we have $\pi = \tilde{\pi}_h$. $\qquad\square$

Let us denote the closure of the $*$-representation $\tilde{\pi}_h$ of $\mathcal{U}_0 \ltimes \mathcal{X}$ by π_h.

Definition 5.4 The closed $*$-representation π_h of $\mathcal{U}_0 \ltimes \mathcal{X}$ defined above is called the *Heisenberg representation* or the *left regular representation* of the cross product $*$-algebra $\mathcal{U}_0 \ltimes \mathcal{X}$ with respect to the $\mathcal{U}_0$-invariant state h.

5.3 A uniqueness theorem for the Heisenberg representation of $\mathcal{U}_{\mathbf{q}}(\mathbf{su_2}) \ltimes \mathcal{O}(\mathbf{SU_q(2)})$

By Lemma 5.2 the $\mathcal{U}_q(su_2)$- module $*$-algebra $\mathcal{O}(SU_q(2))$ has a unique $\mathcal{U}_q(su_2)$-invariant state. Hence the cross product algebra $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ has a unique Heisenberg representation π_h.

Theorem 5.5 (i) *Let π be a closed $*$-representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ such that the restriction of π to $\mathcal{U}_q(su_2)$ is integrable. Then π is unitarily equivalent to a direct sum of Heisenberg representations π_h.*
(ii) *The Heisenberg representation π_h of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ is irreducible and its restriction to $\mathcal{U}_q(su_2)$ is an integrable representation.*

An immediate consequence of this theorem is

Corollary 5.6 *Let π be a closed $*$-representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$. Then π is unitarily equivalent to the Heisenberg representation π_h if and only if π is irreducible and the restriction of π to $\mathcal{U}_q(su_2)$ is integrable.*

Proof of Theorem 5.5. (i): The crucial step of this proof is to show that there exists a non-zero vector v_0 in the domain $\mathcal{D}$ of the representation π such that

$$\pi(E)v_0 = \pi(F)v_0 = 0 \text{ and } \pi(K)v_0 = v_0. \tag{64}$$

By assumption, $\pi \lceil \mathcal{U}_q(su_2)$ is a direct sum of representations $T_{l_i}, i \in I$, where $l_i \in \frac{1}{2}\mathbb{N}_0$. Let us take a non-zero lowest weight vector $v_l \in \mathcal{D}$ of weight $-l$ in the representation space of one of these representations $T_l, l = l_i, i \in I$. That is, we have $\pi(F)v_l = 0$ and $\pi(K)v_l = q^{-l}v_l$. If $l = 0$, then T_l is the trivial subrepresentation of $\mathcal{U}_q(su_2)$ and we are done. If $l \geq 1/2$, then we put

$$w_l := \pi(c)\pi(E)v_l + q^{-l-3/2}[2l]_q \pi(a)v_l,$$
$$w'_l := \pi(d)\pi(E)v_l + q^{-l-3/2}[2l]_q \pi(b)v_l.$$

Using the cross relations of the algebra $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ we derive

$$\pi(F)w_l = \pi(F)w'_l = 0, \pi(K)w_l = q^{-(l-1/2)}w_l, \pi(K)w'_l = q^{-(l-1/2)}w'_l.$$

Further, we have $\pi(a)w'_l - \pi(b)w_l = \pi(ad - qbc)\pi(E)v_l = \pi(E)v_l \neq 0$, because we assumed that $l \geq 1/2$. Hence at least one of the vectors $w_l, w_{l'}$ is non-zero. Thus we have constructed a non-zero lowest weight vector of weight $-(l-1/2)$. Proceeding by induction we obtain after $2l$ steps a non-zero lowest weight vector $v_0 \in \mathcal{D}$ of weight 0, that is, $\pi(F)v_0 = 0$ and $\pi(K)v_0 = v_0$.

From the cross relations of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ we compute $\pi(E)w_{1/2} = \pi(E)w'_{1/2} = 0$, so the vector v_0 satisfies also $\pi(E)v_0 = 0$.

After norming we can assume that $\| v_0 \| = 1$. Obviously, (64) implies

$$\pi(f)v_0 = \varepsilon(f)v_0, f \in \mathcal{U}_q(su_2). \tag{65}$$

Define $h(x) = \langle \pi(x)v_0, v_0 \rangle$ for $x \in \mathcal{O}(SU_q(2))$. Using (4) and (62) we obtain

$$h(x \triangleleft f) = \langle \pi(ad_R(f)x)v_0, v_0 \rangle = \langle \pi(S(f_{(1)}))\pi(x)\pi(f_{(2)})v_0, v_0 \rangle$$
$$= \langle \pi(x)\pi(f_{(2)})v_0, \pi(S(f_{(1)})^*)v_0 \rangle =$$
$$= \langle \pi(x)\varepsilon(f_{(2)})v_0, \varepsilon(S(f_{(1)})^*)v_0 \rangle = \varepsilon(f)\langle \pi(x)v_0, v_0 \rangle = \varepsilon(f)h(x).$$

That is, h is a $\mathcal{U}_q(su_2)$-invariant linear functional on $\mathcal{O}(SU_q(2))$ such that $h(1) = 1$. By Lemma 5.2, h is the Haar state of $\mathcal{O}(SU_q(2))$. Put $\mathcal{D}_0 := \pi(\mathcal{O}(SU_q(2)))v_0$. Let $f \in \mathcal{U}_q(su_2)$ and $x \in \mathcal{O}(SU_q(2))$. By (3) and (65), we have

$$\pi(f)\pi(x)v_0 = \pi(fx)v_0 = \pi((x \triangleleft S^{-1}(f_{(2)}))\pi(f_{(1)})v_0$$
$$= \pi(x \triangleleft S^{-1}(f_{(2)}))\varepsilon(f_{(1)})v_0 = \pi(x \triangleleft S^{-1}(f))v_0, \tag{66}$$

so $\pi(f)$ leaves the domain $\mathcal{D}_0$ invariant. Hence $\mathcal{D}_0$ is invariant under the representation π of the whole algebra $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$. By (65), we have $\pi(f)v_0 = \varepsilon(f)v_0$. Since $h(x) = \langle \pi(x)v_0, v_0 \rangle$ for $x \in \mathcal{O}(SU_q(2))$, the restriction of $\pi \lceil \mathcal{O}(SU_q(2))$ to the domain $\mathcal{D}_0$ is unitarily equivalent to the GNS-representation π_h on $\mathcal{D}_h = \pi_h(\mathcal{O}(SU_q(2)))v_h$. Let π_0 be the closure of the restriction of the $*$-representation π of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ to $\mathcal{D}_0$. By the preceding and the uniqueness assertion of Proposition 5.3, π_0 is unitarily equivalent to the Heisenberg representation π_h.

By assumption the representation $\pi \lceil \mathcal{U}_q(su_2)$ is a direct sum of finite dimensional representations $T_{l_i}, l_i \in \frac{1}{2}\mathbb{N}_0$. Further, the operators $\pi(x)$, $x \in \mathcal{O}(SU_q(2))$, are bounded. From these facts it follows that π decomposes into a direct sum $\pi_0 \oplus \pi'$, where π' satisfies again the assumptions of (i). A standard argument based on Zorn's Lemma gives the assertion.

(ii): Let $t_{ij}^{(l)}$ be the matrix elements of the spin l corepresentation of $SU_q(2)$ and $V_{lj} := \mathrm{Lin}\{t_{ij}^{(l)}; i = -l, \dots, l\}$. By (9) and (63), V_{lj} is invariant under $\pi_h(\mathcal{U}_q(su_2))$, and the subrepresentation of $\mathcal{U}_q(su_2)$ on V_{lj} is unitarily equivalent to T_l, $l \in \frac{1}{2}\mathbb{N}_0$. Since the set $\{t_{ij}^{(l)}; i, j = -l, \dots, l, l \in \frac{1}{2}\mathbb{N}_0\}$ is a vector space basis of $\mathcal{O}(SU_q(2))$ by the Peter-Weyl theorem, $\pi_h \lceil \mathcal{U}_q(su_2)$ is a direct sum of representations $T_l, l \in \frac{1}{2}\mathbb{N}_0$. That is, $\pi_h \lceil \mathcal{U}_q(su_2)$ is integrable.

Finally, we prove that π_h is irreducible. Assume to the contrary that π_h is the direct sum of non-trivial representations π_1 and π_2. Since $\pi_h \lceil \mathcal{U}_q(su_2)$ is integrable as just shown, $\pi_j \lceil \mathcal{U}_q(su_2)$, $j{=}1,2$, is also integrable. By (i), π_j is a direct sum of Heisenberg representations. Hence π_h is unitarily equivalent to a sum of more than two copies of π_h. Then the dimension of the space of $\mathcal{U}_q(su_2)$-invariant vectors in $\pi_h(\mathcal{O}(SU_q(2))v_h$ is larger than one which contradicts the uniqueness of the Haar functional of $SU_q(2)$. $\qquad\square$

5.4 A quantum trace formula for the Haar state of $\mathcal{O}(\mathbf{SU_q(2)})$

If T is a finite dimensional representation of $\mathcal{U}_q(sl_2)$, then the *quantum trace* $\mathrm{Tr}_q(f) := \mathrm{Tr}\, T(K^{-2}f)$, $f \in \mathcal{U}_q(sl_2)$, is an ad_R-invariant linear functional on $\mathcal{U}_q(sl_2)$, that is, $\mathrm{Tr}_q(ad_R(f)g) = \varepsilon(f)\mathrm{Tr}_q(g)$ for $f, g \in \mathcal{U}_q(sl_2)$. This well known fact is based on the trace property and the relation $K^{-2}S(f) = S^{-1}(f)K^{-2}$. For representations on infinite dimensional Hilbert spaces the quantum trace in the above form does not make sense. In this subsection and in Subsection 6.3 below we develop two variants of the quantum trace and describe the Haar state of $SU_q(2)$ in this manner.

For $z \in \mathbb{C}$, $\mathrm{Re}\, z > 1$, we define a holomorphic function

$$\zeta(z) = (q^{-1} - q)^{2z-1} \sum_{n=1}^{\infty} n(q^{-\frac{n}{2}} - q^{\frac{n}{2}})^{-2z}(q^{-n} - q^n).$$

Note that $q^{-1} - q > 0$ and $q^{-\frac{n}{2}} - q^{\frac{n}{2}} > 0$ for $n \in \mathbb{N}$, since $0 < q < 1$. Let $\mathcal{C}$ denote the closure of the operator $\pi_h(C_q)$, where C_q is the Casimir element (59) of $\mathcal{U}_q(su_2)$. Recall that h is the Haar state of $\mathcal{O}(SU_q(2))$.

Theorem 5.7 *If $z \in \mathbb{C}$, Re $z > 1$, and $x \in \mathcal{O}(SU_q(2))$, then the closure of the operator $\mathcal{C}^{-z}\pi_h(K^{-2}x)$ is of trace class and we have*

$$h(x) = \zeta(z)^{-1}\, \mathrm{Tr}\, \overline{\mathcal{C}^{-z}\pi_h(K^{-2}x)}\ . \tag{67}$$

Proof. The representation $\pi_h \lceil \mathcal{U}_q(su_2)$ is the direct sum of representations $(2l + 1)T_l$, $l \in \frac{1}{2}\mathbb{N}_0$. Hence, by (58) and (60), the operators $\overline{\pi_h(K)}$, $|\overline{\pi_h(E)}|$, $|\overline{\pi_h(F)}|$, and $\mathcal{C} = \overline{\pi_h(C_q)}$ have a common orthonormal basis of eigenvectors with eigenvalues q^j, $\alpha_{j+1,l}$, α_{jl}, $j = -l, \ldots, l$, each with multiplicity $2l+1$, and $[l + 1/2]_q^2$ with multiplicity $(2l+1)^2$, respectively. Since $|\alpha_{jl}| \leq \mathrm{const.}\, q^{-l}$, $[l +$

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$1/2]_q^{-2} \leq$ const. q^{2l} and $\pi_h(x)$, $x \in \mathcal{O}(SU_q(2))$, is bounded, it follows that the closures of the operators $\mathcal{C}^{-z}\pi_h(K^{-2}x)$, $\mathcal{C}^{-z}\pi_h(K^{-1}Ex)$ and $\mathcal{C}^{-z}\pi_h(K^{-1}Fx)$ are of trace class when Re $z > 1$. Thus, $h_z(x) := \text{Tr } \overline{\mathcal{C}^{-z}\pi_h(K^{-2}x)}$ is well defined for $x \in \mathcal{O}(SU_q(2))$ and $z \in \mathbb{C}$, Re $z > 1$. By the cross relations of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$, each element $xK^{-1}E$ is of the form $K^{-1}Ex' + K^{-2}x''$ with $x', x'' \in \mathcal{O}(SU_q(2))$. Hence the closures of $\mathcal{C}^{-z}\pi_h(xK^{-1}E)$ and likewise of $\mathcal{C}^{-z}\pi_h(xK^{-1}F)$ are also of trace class.

Suppose $z \in \mathbb{C}$, Re $z > 1$, and $x \in \mathcal{O}(SU_q(2))$. There is $y \in \mathcal{O}(SU_q(2))$ such that $xK = Ky$. Using the facts of the preceding paragraph we conclude

$$\text{Tr } \overline{\mathcal{C}^{-2z}\pi_h(yK^{-1}E)} = \text{Tr } \overline{\mathcal{C}^{-z}\mathcal{C}^{-z}\pi_h(yK^{-1}E)}$$
$$= \text{Tr } \overline{\mathcal{C}^{-z}\pi_h(yK^{-1}E)\mathcal{C}^{-z}} = \text{Tr } \overline{\mathcal{C}^{-z}\pi_h(y)\pi_h(K^{-1}E)\mathcal{C}^{-z}}$$
$$= \text{Tr } \overline{\pi_h(K^{-1}E)\mathcal{C}^{-z}\mathcal{C}^{-z}\pi_h(y)} = \text{Tr } \overline{\mathcal{C}^{-2z}\pi_h(K^{-1}Ey)}.$$

From (4) and the latter formula we get

$$h_{2z}(x \triangleleft E) = h_{2z}(ad_R(E)x) = \text{Tr } \overline{\mathcal{C}^{-2z}\pi_h(K^{-2}(KxE - qExK))}$$
$$= \text{Tr } \overline{\mathcal{C}^{-2z}\pi_h(yK^{-1}E - K^{-1}Ey)}$$
$$= \text{Tr } \overline{\mathcal{C}^{-2z}\pi_h(yK^{-1}E)} - \text{Tr } \overline{\mathcal{C}^{-2z}\pi_h(K^{-1}Ey)} = 0.$$

Similarly, one shows that $h_{2z}(x \triangleleft F) = 0$ and $h_{2z}(x \triangleleft K^{\pm 1}) = h_{2z}(x)$. Hence $h_{2z}(x \triangleleft f) = \varepsilon(f)h_{2z}(x)$ for $f \in \mathcal{U}_q(su_2)$. Thus h_{2z} is a $\mathcal{U}_q(su_2)$-invariant linear functional on $\mathcal{O}(SU_q(2))$. By Lemma 5.2, $h_{2z}(1)^{-1}h_{2z}$ is the Haar state h of $\mathcal{O}(SU_q(2))$.

Now we compute $h_z(1)$ for $z \in \mathbb{C}$, Re $z > 1$. The trace of the restriction of the operator $\mathcal{C}^{-z}\pi_h(K^{-2})$ to the invariant subspace V_l is

$$\sum_{i,j=-l}^{l} [l + 1/2]_q^{-2z}q^{-2j} = (2l+1)[l + 1/2]_q^{-2z}[2l + 1]_q.$$

Summing over $l \in \frac{1}{2}\mathbb{N}_0$ or equivalently over $n = 2l+1 \in \mathbb{N}$, we get

$$h_z(1) = \text{Tr } \overline{\mathcal{C}^{-z}\pi_h(K^{-2})} = \sum_{n=1}^{\infty} n[n/2]_q^{-2z}[n]_q = \zeta(z).$$

By the preceding we have proved that $h = h_z(1)^{-1}h_z = \zeta(z)^{-1}h_z$ for $z \in \mathbb{C}$, Re $z > 2$. Let $x \in \mathcal{O}(SU_q(2))$. Then, $\zeta(z)h(x)$ and $h_z(x)$ are holomorphic functions of $z \in \mathbb{C}$, Re $z > 1$. As just shown, the two functions are equal for Re $z > 2$. Hence they coincide also for Re $z > 1$. □

The function $\zeta(z)$ is called the *zeta function* of the quantum group $SU_q(2)$. The functional h_z and so formula (67) can be rewritten as

$$h_z(x) = \mathrm{Tr}\ \overline{\pi_h(x)}\ \overline{\pi_h(K^{-2})\pi_h(C_q)^{-z}} = \mathrm{Tr}\ \overline{\pi_h(C_q)^{-z}\pi_h(K^{-2})}\ \overline{\pi_h(x)}.$$

Note that the operators $\overline{\pi_h(K^{-2})\pi_h(C_q)^{-z}}$ and $\overline{\pi_h(C_q)^{-z}\pi_h(K^{-2})}$ are of trace class if Re $z > 1$ and $\overline{\pi_h(x)}, x \in \mathcal{O}(SU_q(2))$, is bounded.

6 Representations of cross product $*$-algebras

We now develop the second approach to representations of cross product algebras. That is, we begin with a $*$-representation of one of the coordinate $*$-algebras $\mathcal{O}(SU_q(2))$, $\hat{\mathcal{O}}(\mathbb{C}_q^2)$, and $\mathcal{O}(\mathbb{R}_q^3)$ as described in 4.2 and try to complete it to a $*$-representation of the cross product $*$-algebra. In doing so we require that certain relations derived by formal algebraic manipulations hold in the operator-theoretic sense in the Hilbert space. For notational simplicity we suppress the representation and write x for $\pi(x)$ when no confusion can arise. Recall that $0 < q < 1$, $p > 0$ and $p \neq 1$.

6.1 Representations of the $*$-algebra $\mathcal{U_0} \ltimes \mathcal{O}(\mathbf{SU_q(2)})$

Suppose we have a $*$-representation of $\mathcal{U}_0 \ltimes \mathcal{O}(SU_q(2))$ on a Hilbert space such that its restriction to $\mathcal{O}(SU_q(2))$ is of the form described in 4.2.

We assume that there exist dense linear subspaces $\mathcal{E}$ and $\mathcal{D}_0$ of $\mathcal{G}$ and $\mathcal{H}_0$, respectively, such that $v\mathcal{E} = \mathcal{E}, w\mathcal{D}_0 = \mathcal{D}_0$ and $\mathcal{E} \oplus \mathcal{D}$ is invariant under the X_j, $j = 0, 1, 2$, where $\mathcal{D} = \mathrm{Lin}\{\eta_n; \eta \in \mathcal{D}_0,\ n \in \mathbb{N}_0\}$.

1. Step: First we show that $\mathcal{G} = \{0\}$. Since $X_2^* = X_0$ and $b = c = 0$ on $\mathcal{G}$, it follows from the relations $bX_2 = q^{-1}X_2b$ and $cX_0 = qX_0c$ that X_2 leaves the subspace $\mathcal{G}$ invariant. Thus, $cX_2\varphi = 0$ for $\varphi \in \mathcal{E}$. Therefore, since $cX_2 = qX_2c + a$, we obtain $a\varphi = 0$ for $\varphi \in \mathcal{E}$. Because $a = v$ is unitary on $\mathcal{G}$ and $\mathcal{E}$ is dense in $\mathcal{G}$, the latter implies that $\mathcal{G} = \{0\}$.

Since $\mathcal{H}_0 = \ker a$, it follows from the relation $aY_1 = q^{-2}Y_1a$ that Y_1 leaves $\mathcal{H}_0$ invariant, so there is a symmetric operator G_0 on $\mathcal{H}_0$ such that $Y_1\eta_0 = G_0\eta_0, \eta_0 \in \mathcal{D}_0$. From $d^nY_1 = q^{2n}Y_1d^n$ we conclude that $Y_1\eta_n = q^{-2n}G_0\eta_n$. The relation $cY_1 = q^2Y_1c$ implies that $w^*G_0\eta_n = q^2G_0w^*\eta_n$.

From the relations $aX_2 = q^{-1}X_2a$ and $dX_2 = qX_2d + b$ it follows by induction on n that X_2 maps $\mathcal{H}_n$ into $\mathcal{H}_n + \mathcal{H}_{n-1}$. Hence there exist linear operators

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T_n and R_n on the Hilbert space $\mathcal{H}_0$ such that $X_2\eta_n = T_n\eta_n + R_n\eta_{n-1}, \eta \in \mathcal{D}_0$. Inserting this into the relation $aX_2 = q^{-1}X_2a$ shows that $T_n = q^{-1}T_{n-1}$. Thus we obtain $T_n = q^{-n}T_0$. Comparing the $(n-1)$th components of the relation $bcX_2 = X_2bc + ba$ gives $R_n = q^{-n}\lambda_n\lambda^{-1}w$. Since $bX_2 = q^{-1}X_2b$, we obtain $wT_0 = q^{-1}T_0w$.

We have not yet used the commutation relations (34)–(36) of the generators X_0, X_2, X_1. Equation (35) is equivalent to $Y_1X_2 = q^4X_2Y_1$. Using the formulas for Y_1 and X_2 obtained above we get $G_0T_0 = q^4T_0G_0$. Since $X_0 = X_2^*$, we have $X_0\eta_n = T_n^*\eta_n + R_{n+1}^*\eta_{n+1}, \eta \in \mathcal{D}_0$. Inserting the above expressions of X_0, X_2, T_n, R_n into relation (36) yields the equation $qT_0T_0^* - q^{-1}T_0^*T_0 = \lambda^{-1}G_0$.

Summarizing the first step, we have shown that X_0, X_2, Y_1 act as

$$X_2\eta_n = q^{-n}T_0\eta_n + q^{-n}\lambda_n\lambda^{-1}w\eta_{n-1}, \tag{68}$$

$$X_0\eta_n = q^{-n}T_0^*\eta_n + q^{-n-1}\lambda_{n+1}\lambda^{-1}w^*\eta_{n+1}, \tag{69}$$

$$Y_1\eta_n = q^{-2n}G_0\eta_n, \eta \in \mathcal{D}_0, \tag{70}$$

where the operators T_0, G_0, w satisfy the consistency conditions

$$wG_0w^* = q^{-2}G_0, \ G_0T_0 = q^4T_0G_0, \ wT_0w^* = q^{-1}T_0, \tag{71}$$

$$qT_0T_0^* - q^{-1}T_0^*T_0 = \lambda^{-1}G_0 \tag{72}$$

on the domain $\mathcal{D}_0$ of $\mathcal{H}_0$. Conversely, if operators T_0, G_0, w on the Hilbert space $\mathcal{H}_0$ are given such that G_0 is symmetric, w is unitary and relations (71) and (72) are fulfilled on a dense domain $\mathcal{D}_0$ of $\mathcal{H}_0$ which is invariant for the operators T_0, T_0^*, G_0, w, and w^*, then the formulas (50), (68)–(70) define a $*$-representation of $\mathcal{U}_0 \ltimes \mathcal{O}(SU_q(2))$. Indeed, by straightforward computations it can be checked that the defining relations of $\mathcal{U}_0 \ltimes \mathcal{O}(SU_q(2))$ are satisfied.

2. Step: Next we analyze the triple of operators T_0, G_0, w on the Hilbert space $\mathcal{H}_0$ satisfying the consistency conditions (71)–(72).

We suppose that the closure of the operator Y_1 is self-adjoint. By the above assumption on the domain, we can suppose that G_0 is self-adjoint. Let $g(\lambda), \lambda \in \mathbb{R}$, denote the spectral projections of G_0. The Hilbert space $\mathcal{H}_0$ decomposes into a direct sum

$$\mathcal{H}_0 = g((-\infty, 0))\mathcal{H}_0 \oplus g(\{0\})\mathcal{H}_0 \oplus g((0, \infty))\mathcal{H}_0 \tag{73}$$

of reducing subspaces of G_0 where $G_0 < 0$, $G_0 = 0$ and $G_0 > 0$, respectively. Since $wG_0w^* = q^{-2}G_0$, the direct sum (73) reduces w as well. Since $G_0T_0 =$

$q^4 T_0 G_0$, we assume that (73) reduces also T_0. Thus we are lead to study relations (71)–(72) in the three cases $G_0 = 0$, $G_0 < 0$, $G_0 > 0$ separately.

Case I. $G_0 = 0$

Then relations (71)–(72) read $w T_0 w^* = q^{-1} T_0$ and $q^2 T_0 T_0^* = T_0^* T_0$. Obviously, there is the trivial representation where $T_0 = 0$ and w is arbitrary. Since $\ker T_0 = \ker T_0^*$ is reducing for T_0 and w, we can consider the cases $T_0 = 0$ and $\ker T_0 = \{0\}$ separately. Assume now that $\ker T_0 = \{0\}$ and let $T_0 = v_0 |T_0|$ be the polar decomposition of T_0. Since $\ker T_0 = \ker T_0^* = \{0\}$, v_0 is unitary. From $w T_0 w^* = q^{-1} T_0$ we get $w |T_0| w^* = q^{-1} |T_0|$. The relation $q^2 T_0 T_0^* = T_0^* T_0$ implies that $q^2 v_0 |T_0|^2 v_0^* = |T_0|^2$ and so $q v_0 |T_0| v_0^* = |T_0|$. Hence we have $w^* v_0 |T_0| v_0^* w = q^{-1} w^* |T_0| w = |T_0|$. Using the preceding relations and the fact that w and v_0 are unitary, we get $w^* T_0 = w^* v_0 |T_0| = |T_0| w^* v_0 = q T_0 w^* = q v_0 |T_0| w^* = v_0 w^* |T_0|$. Since $\ker |T_0| = \ker T_0 = \{0\}$, we obtain $w^* v_0 = v_0 w^*$ and so $w v_0 = v_0 w$.

Since w is unitary, by Lemma 4.2(i) the relation $w |T_0| w^* = q^{-1} |T_0|$ leads to a representation $w \zeta_k = \zeta_{k+1}$, $|T_0| \zeta_k = q^k A_{00} \zeta_k$ on $\mathcal{H}_0 = \oplus_{n=-\infty}^{\infty} \mathcal{H}_{0k}$, where $\mathcal{H}_{0k} = \mathcal{H}_{00}$ and A_{00} is a self-adjoint operator on the Hilbert space $\mathcal{H}_{00}$ such that $\sigma(A_{00}) \sqsubseteq (q, 1]$. The operator $w^* v_0$ commutes with $|T_0|$ and so with the spectral projections of $|T_0|$. Since $\sigma(A_{00}) \sqsubseteq (q, 1]$, $w^* v_0$ leaves each space $\mathcal{H}_{0k}$ invariant. Therefore, since $w v_0 = v_0 w$, there is a unitary operator v_{00} on $\mathcal{H}_{0k}$ such that $v_0 \zeta_k = v_{00} \zeta_{k+1}$, $k \in \mathbb{Z}$. Using again the relation $q^2 T_0 T_0^* = T_0^* T_0$, it follows that the operator $N := v_{00} A_{00}$ on $\mathcal{H}_{00}$ is normal. Thus we have $T_0 \zeta_k = q^k N \zeta_{k+1}$ with N normal. This completes the treatment of Case I.

Next we treat the cases $G_0 < 0$ and $G_0 > 0$. Let us set $G_0 = \delta H_0^2$ and $H_0 := \epsilon |G_0|^{1/2}$, where $\epsilon, \delta \in \{1, -1\}$. Since $G_0 T_0 = q^4 T_0 G_0$, it is natural to assume that $H_0 T_0 = q^2 T_0 H_0$. (It can be shown that the relation $H_0 T_0 = -q^2 T_0 H_0$ does not have a non-trivial solution for $H_0 > 0$.) Set $S_0 := H_0^{-1} T_0$. We rewrite the consistency conditions in terms of H_0, S_0, w by formal (!) algebraic manipulations. Since $S_0 = H_0^{-1} T_0 = q^{-2} T_0 H_0^{-1}$ and so $S_0^* = T_0^* H_0^{-1} = q^{-2} H_0^{-1} T_0^*$, (72) is formally equivalent to

$$S_0 S_0^* - q^2 S_0^* S_0 = -\delta (1 - q^2)^{-1} , \qquad (74)$$

where $\delta \in \{1, -1\}$. The three relations (71) can be rewritten as

$$w H_0 w^* = q^{-1} H_0, \ \ H_0 S_0 = q^2 S_0 H_0, \ \ w S_0 w^* = S_0 . \qquad (75)$$

We now solve the relations (74) and (75) in a rigorous manner.

Case II. $G_0 < 0$ $(\delta = -1)$

Since $0 < q < 1$ and $-\delta(1-q^2)^{-1} > 0$, relation (74) has three series of representations by Lemma 4.3. Let us begin with the *Fock representation*. Then we have

$$S_0\zeta_k = (1-q^2)^{-1}\lambda_k\zeta_{k-1}, \ S_0^*\zeta_k = (1-q^2)^{-1}\lambda_{k+1}\zeta_{k+1} \tag{76}$$

acting on the direct sum Hilbert space $\mathcal{H}_0 = \oplus_{k=0}^{\infty}\mathcal{H}_{0k}$, where $\mathcal{H}_{0k} = \mathcal{H}_{00}$. Since $wS_0w^* = S_0$ by (75), w commutes with the spectral projections of $S_0S_0^*$ and so w leaves each space $\mathcal{H}_{0k}$ invariant. Hence, by (76), the relation $wS_0w^* = S_0$ implies that there is a unitary w_0 on $\mathcal{H}_{00}$ such that $w\zeta_k = w_0\zeta_k$. The relation $H_0S_0 = q^2S_0H_0$ yields $H_0S_0^*S_0 = S_0^*S_0H_0$. We assume that the commuting self-adjoint operators H_0 and $S_0^*S_0$ *strongly* commute. Then H_0 commutes with the spectral projections of $S_0^*S_0$, so H_0 leaves $\mathcal{H}_{0k}$, $k \in \mathbb{N}_0$, invariant. Hence there are positive self-adjoint operators H_{0k} on $\mathcal{H}_{0k}$ such that $H_0\zeta_k = \epsilon H_{0k}\zeta_k$, $\epsilon \in \{1, -1\}$. From $H_0S_0 = q^2S_0H_0$ we conclude that $H_{0k} = q^{-2k}H_{00}$. Finally, the relation $wH_0w^* = q^{-1}H_0$ implies that $w_0H_{00}w_0^* = q^{-1}H_{00}$ holds on the Hilbert space $\mathcal{H}_{00}$. Conversely, if the latter is true, then the preceding formulas define operators H_0, S_0, w fulfilling (74)–(75). The representations of the relation $w_0H_{00}w_0^* = q^{-1}H_{00}$ are derived from Lemma 4.2(i).

Now we take the *second series of representations* of (74). There is a self-adjoint operator A_{00} on a Hilbert space $\mathcal{H}_{00}$ such that $\sigma(A_{00}) \sqsubseteq (q^2, 1]$ and

$$S_0\zeta_k = (1-q^2)^{-1}\alpha_k(A_{00})\zeta_{k-1}, \ S_0^*\zeta_k = (1-q^2)^{-1}\alpha_{k+1}(A_{00})\zeta_{k+1}$$

acting on the Hilbert space $\mathcal{H}_0 = \oplus_{k=-\infty}^{\infty}\mathcal{H}_{0k}$, where $\mathcal{H}_{0k} = \mathcal{H}_{00}$. Arguing as in the preceding paragraph, we conclude that there exist a unitary operator w_0 and a positive self-adjoint operator H_{00} on the Hilbert space $\mathcal{H}_{00}$ satisfying the relations $A_{00}H_{00} = H_{00}A_{00}, w_0A_{00}w_0^* = A_{00}$ and $w_0H_{00}w_0^* = q^{-1}H_{00}$ such that $w\zeta_k = w_0\zeta_k$ and $H_0\zeta_k = q^{-2k}\epsilon H_{00}\zeta_k$ for $\zeta \in \mathcal{H}_{00}$, $k \in \mathbb{Z}$. Conversely, if the latter holds, then relations (74)–(75) are satisfied.

Finally we turn to the *third series of represengions* of (74). Then there is a unitary operator v_0 on $\mathcal{H}_0$ such that $S_0\eta = (1-q^2)^{-1}v_0\eta$, $\eta \in \mathcal{H}_0$. The relation $wH_0w^* = q^{-1}H_0$ leads to $w|H_0|w^* = q^{-1}|H_0|$. Recall that $wH_0w^* = q^{-1}H_0$ by (75), $H_0 = \epsilon|H_0|$, where $\epsilon \in \{1, -1\}$, and $\ker H_0 = \{0\}$. By Lemma 4.2(i), there is a self-adjoint operator H_{00} on a Hilbert space $\mathcal{H}_{00}$ such that $\sigma(H_{00}) \sqsubseteq (q, 1]$ and $H_0\zeta_k = q^k\epsilon H_{00}\zeta_k$, $w\zeta_k = \zeta_{k+1}$ on $\mathcal{H}_0 = \oplus_{k=-\infty}^{\infty}\mathcal{H}_{0k}$, $\mathcal{H}_{0k} = \mathcal{H}_{00}$. From $H_0S_0 = q^2S_0H_0$ by (75) we obtain $H_0v_0 = q^2v_0H_0$, hence

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$v_0 w^{*2}$ commutes with H_0. From this and the relation $v_0 w = w v_0$ by (75), it follows that there is a unitary v_{00} on $\mathcal{H}_{00}$ such that $v_{00} H_{00} v_{00}^* = H_{00}$ and $v_0 \zeta_k = v_{00} \zeta_{k+2}, k \in \mathbb{Z}$. Conversely, the latter gives indeed a representation of relations (74)–(75).

Case III. $G_0 > 0$ $(\delta = 1)$

Then relation (74) reads

$$S_0^* S_0 - q^{-2} S_0 S_0^* = q^{-2}(1 - q^2)^{-1} . \tag{77}$$

Since $q < 1$, relation (77) has only the Fock representation by Lemma 4.3. Thus there is a Hilbert space $\mathcal{H}_{00}$ such that $\mathcal{H}_0 = \oplus_{k=0}^{\infty} \mathcal{H}_{0k}, \mathcal{H}_{0k} = \mathcal{H}_{00}$, and

$$S_0 \zeta_k = (1-q^2)^{-1} q^{-k-1} \lambda_{k+1} \zeta_{k+1}, \ S_0^* \zeta_k = (1-q^2)^{-1} q^{-k} \lambda_k \zeta_{k-1} .$$

The other consistency relations (75) are treated in the same manner as for the Fock representation in Case II. This completes the treatment of Step 2.

Now we bring all the above considerations together. First we insert the representation of the relation $w_0 H_{00} w_0^* = q^{-1} H_{00}$ from Lemma 4.2(i). Then we put the expressions for the operators $S_0 = H_0^{-1} T_0, H_0$ and w derived in the preceding paragraphs into formulas (68)–(70). In doing so, we finally obtain the following list of $*$-representations of the $*$-algebra $\mathcal{U}_0 \ltimes \mathcal{O}(SU_q(2))$:

$(I.1)_w:$ $X_2 \eta_n = \lambda^{-1} q^{-n} \lambda_n w \eta_{n-1},$

   $X_0 \eta_n = \lambda^{-1} q^{-n-1} \lambda_{n+1} w^* \eta_{n+1},$

   $Y_1 \eta_n = 0$ on $\mathcal{H} = \overset{\infty}{\underset{n=0}{\oplus}} \mathcal{H}_n, \ \mathcal{H}_n = \mathcal{K}.$

$(I.2)_N:$ $X_2 \eta_{nk} = q^{-n+k} N \eta_{n,k+1} + \lambda^{-1} q^{-n} \lambda_n \eta_{n-1,k+1},$

   $X_0 \eta_{nk} = q^{-n+k-1} N^* \eta_{n,k-1} + \lambda^{-1} q^{-n-1} \lambda_{n+1} \eta_{n+1,k-1},$

   $Y_1 \eta_{nk} = 0$ on $\mathcal{H} = \overset{\infty}{\underset{n=0}{\oplus}} \overset{\infty}{\underset{k=-\infty}{\oplus}} \mathcal{H}_{nk}, \ \mathcal{H}_{nk} = \mathcal{K}.$

$(II.1)_{H,\epsilon}:$ $X_2 \eta_{nkl} = q^{-n-2k+l+1} \lambda_k \lambda^{-1} \epsilon H \eta_{n,k-1,l} + q^{-n} \lambda^{-1} \lambda_n \eta_{n-1,k,l+1},$

   $X_0 \eta_{nkl} = q^{-n-2k+l-1} \lambda_{k+1} \lambda^{-1} \epsilon H \eta_{n,k+1,l} + q^{-n-1} \lambda_{n+1} \lambda^{-1} \eta_{n+1,k,l-1},$

   $Y_1 \eta_{nkl} = -q^{-2n-4k+2l} H^2 \eta_{nkl}$ on $\mathcal{H} = \overset{\infty}{\underset{n,k=0}{\oplus}} \overset{\infty}{\underset{l=-\infty}{\oplus}} \mathcal{H}_{nkl}, \ \mathcal{H}_{nkl} = \mathcal{K}.$

$(II.2)_{A,H,\epsilon}$: $X_2\eta_{nkl}=q^{-n-2k+l+1}\alpha_k(A)\lambda^{-1}\epsilon H\eta_{n,k-1,l} + q^{-n}\lambda_n\lambda^{-1}\eta_{n-1,k,l+1},$

$\qquad X_0\eta_{nkl}=q^{-n-2k+l-1}\alpha_{k+1}(A)\lambda^{-1}\epsilon H\eta_{n,k+1,l}+q^{-n-1}\lambda_{n+1}\lambda^{-1}\eta_{n+1,k,l-1},$

$\qquad Y_1\eta_{nkl}=-q^{-2n-4k+2l}H^2\eta_{nkl}$ on $\mathcal{H}=\overset{\infty}{\underset{n=0}{\oplus}}\ \overset{\infty}{\underset{k,l=-\infty}{\oplus}}\mathcal{H}_{nkl}, \mathcal{H}_{nkl}=\mathcal{K}.$

$(II.3)_{H,v,\epsilon}$: $X_2\eta_{nk}=q^{-n+k+1}\lambda^{-1}\epsilon Hv\eta_{n,k+2} + q^{-n}\lambda_n\lambda^{-1}\eta_{n-1,k+1},$

$\qquad X_0\eta_{nk}=q^{-n+k-1}\lambda^{-1}\epsilon Hv^*\eta_{n,k-2} + q^{-n-1}\lambda_{n+1}\lambda^{-1}\eta_{n+1,k-1},$

$\qquad Y_1\eta_{nk}=-q^{-2n+2k}H^2\eta_{nk}$ on $\mathcal{H}=\overset{\infty}{\underset{n=0}{\oplus}}\ \overset{\infty}{\underset{k=-\infty}{\oplus}}\mathcal{H}_{nk},\ \mathcal{H}_{nk}=\mathcal{K}.$

$(III)_{H,\epsilon}$: $X_2\eta_{nkl}=q^{-n+k+l}\lambda_{k+1}\lambda^{-1}\epsilon H\eta_{n,k+1,l} + q^{-n}\lambda_n\lambda^{-1}\eta_{n-1,k,l+1},$

$\qquad X_0\eta_{nkl}=q^{-n+k+l-1}\lambda_k\lambda^{-1}\epsilon H\eta_{n,k-1,l} + q^{-n-1}\lambda_{n+1}\lambda^{-1}\eta_{n+1,k,l-1},$

$\qquad Y_1\eta_{nkl}=q^{-2n+4k+2l}H^2\eta_{nkl}$ on $\mathcal{H}=\overset{\infty}{\underset{n,k=0}{\oplus}}\ \overset{\infty}{\underset{l=-\infty}{\oplus}}\mathcal{H}_{nkl},\ \mathcal{H}_{nkl}=\mathcal{K}.$

Here $\epsilon \in \{1,-1\}$, N is a normal operator, A and H are self-adjoint operators, and w and v are unitaries acting on a Hilbert space $\mathcal{K}$ such that $\sigma(|N|) \sqsubseteq (q,1], \sigma(A) \sqsubseteq (q^2,1]$ and $\sigma(H) \sqsubseteq (q,1]$. Further, $AH = HA$ in $(II.2)_{A,H,\epsilon}$ and $vH = Hv$ in $(II.3)_{H,v,\epsilon}$. The series $(I),(II)$ and (III) correspond to the three cases I,II,III discussed above. To complete the picture, we state the actions of the generators a,b,c,d:

$(I.1)_w$:
$a\eta_n=\lambda_n\eta_{n-1},\ d\eta_n=\lambda_{n+1}\eta_{n+1},\ b\eta_n=q^{n+1}w\eta_n,\ c\eta_n=-q^nw^*\eta_n.$
$(I.2)_N, (II.3)_{H,v,\epsilon}$:
$a\eta_{nk}=\lambda_n\eta_{n-1,k},\ d\eta_{nk}=\lambda_{n+1}\eta_{n+1,k},\ b\eta_{nk} = q^{n+1}\eta_{n,k+1},\ c\eta_{nk}= -\ q^n\eta_{n,k-1}$.
$(II.1)_{H,\epsilon},\ (II.2)_{A,H,\epsilon},\ (III)_{H,\epsilon}$:
$a\eta_{nkl}=\lambda_n\eta_{n-1,kl}, d\eta_{nkl}=\lambda_{n+1}\eta_{n+1,kl}, b\eta_{nkl}=q^{n+1}\eta_{nk,l+1}, c\eta_{nkl}=-q^n\eta_{nk,l-1}.$ (78)

A representation on this list is irreducible if and only if the Hilbert space $\mathcal{K}$ has dimension one. In this case the parameters N,A,H,w,v are complex numbers such that $|N| \in (q,1]$, $A \in (q^2,1]$, $H \in (q,1]$, $|w| = 1$, and $|v| = 1$. Representations corresponding to different sets of parameters N,A,H,w,v,ϵ, respectively, or belonging to different series are not unitarily equivalent.

Recall that the generator Y_1 of $\mathcal{U}_0$ corresponds to the element K^4 of $\mathcal{U}_q(su_2)$. Hence only the representations $(III)_{H,\epsilon}$ of $\mathcal{U}_0 \ltimes \mathcal{O}(SU_q(2))$ extend to $*$-representations of the larger $*$-algebra $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$.

6.2 Representations of the ∗-algebra $\mathcal{U}_q(\mathbf{su_2}) \ltimes \mathcal{O}(\mathbf{SU_q(2)})$

The procedure is similar to that in the preceding subsection. Let us suppose that we have a ∗-representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ on a Hilbert space $\mathcal{H}$ such that its restriction to the ∗-subalgebra $\mathcal{O}(SU_q(2))$ is of the form given in 4.2.

1. Step: As in the preceding subsection we conclude that $\mathcal{G} = \{0\}$ and we assume that the operator K is essentially self-adjoint. From the relation $aK = q^{-1/2}Ka$ and $dK = q^{1/2}Kd$ it follows that there is an invertible self-adjoint operator K_0 on $\mathcal{H}_0$ such that $K\eta_n = q^{-n/2}K_0\eta_n$. Since $bK = q^{-1/2}Kb$, we have $wK_0\eta_n = q^{-1/2}K_0 w\eta_n$. The relation $dE = q^{1/2}Ed + K^{-1}b$ implies that E is of the form $E\eta_n = T_n\eta_n + R_n\eta_{n-1}, \eta \in \mathcal{D}_0$, where T_n and R_n are linear operators on $\mathcal{H}_0$. From $aE = q^{-1/2}Ea$ we get $T_n = q^{-n/2}T_0$ and from $daE = Eda + q^{-1/2}K^{-1}ba$ we derive that $R_n = q^{-n/2}\lambda_n\lambda^{-1}K_0^{-1}w$. The relations $bE = q^{-1/2}Eb$ and $KE = qEK$ yield $wT_0 = q^{-1/2}T_0w$ and $K_0T_0 = qT_0K_0$, respectively. The defining relation $EF - FE \equiv EE^* - E^*E = \lambda^{-1}(K^2 - K^{-2})$ leads to $T_0T_0^* - T_0^*T_0 = \lambda^{-1}K_0^2$. We rewrite this in terms of $S_0 := K_0^{-1}T_0$.

The operators E, F, K act as

$$E\eta_n = q^{-n/2}K_0S_0\eta_n + q^{-n/2}\lambda_n\lambda^{-1}K_0^{-1}w\eta_{n-1} \ , \tag{79}$$

$$F\eta_n = q^{-n/2}S_0^*K_0\eta_n + q^{-(n+1)/2}\lambda_{n+1}\lambda^{-1}w^*K_0^{-1}\eta_{n+1} \ , \tag{80}$$

$$K\eta_n = q^{-n/2}K_0\eta_n \ , \tag{81}$$

where the operators S_0, K_0, w satisfy the consistency conditions

$$S_0^*S_0 - q^{-2}S_0S_0^* = (q(1 - q^2))^{-1} \ , \tag{82}$$

$$wK_0w^* = q^{-1/2}K_0, \ K_0S_0 = qS_0K_0, \ wS_0w^* = S_0 \ . \tag{83}$$

Conversely, if (82) and (83) are fulfilled, then the operators E, F, K defined by (79)–(81) satisfy the defining relations of the algebra $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$.

2. Step: Next we investigate the consistency relations (82) and (83). Since $0 < q < 1$, by Lemma 4.3 relation (82) has only the Fock representation (see also Case III in 6.1). Hence there is a Hilbert space $\mathcal{H}_{00}$ such that $\mathcal{H}_0 = \oplus_{k=0}^{\infty} \mathcal{H}_{0k}, \mathcal{H}_{0k} = \mathcal{H}_{00}$, and

$$S_0\zeta_k = q^{-1/2}\lambda^{-1}q^{-k-1}\lambda_{k+1}\zeta_{k+1}, \ S_0^*\zeta_k = q^{-1/2}\lambda^{-1}q^{-k}\lambda_k\zeta_{k-1} \ .$$

Arguing as in the preceding subsection, it follows from (83) that there are a unitary operator w_0 and an invertible self-adjoint operator K_{00} on $\mathcal{H}_{00}$ satisfying $w_0K_{00}w_0^* = q^{-1/2}K_{00}$ such that $w\zeta_k = w_0\zeta_k$ and $K_0\zeta_k = q^kK_{00}\zeta_k, \zeta \in \mathcal{H}_{00}$.

Using the representations of the relation $w_0 K_{00} w_0^* = q^{-1/2} K_{00}$ from Lemma 4.2(i) and inserting the preceding into the formulas (79)–(81) we obtain the following series of $*$-representations of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$:

$(I)_{H,\epsilon}$:
$$E\eta_{nkl} = q^{-(n-l+1)/2}\lambda_{k+1}\lambda^{-1}\epsilon H\eta_{n,k+1,l} + q^{-(n+2k+l+1)/2}\lambda_n\lambda^{-1}\epsilon H^{-1}\eta_{n-1,k,l+1} ,$$
$$F\eta_{nkl} = q^{-(n-l+1)/2}\lambda_k\lambda^{-1}\epsilon H\eta_{n,k-1,l} + q^{-(n+2k+l+1)/2}\lambda_{n+1}\lambda^{-1}\epsilon H^{-1}\eta_{n+1,k,l-1},$$

$$K\eta_{nkl} = q^{(-n+2k+l)/2}\epsilon H\eta_{nkl} .$$

Here $\epsilon \in \{1, -1\}$ and H is a self-adjoint operator on a Hilbert space $\mathcal{K}$ such that $\sigma(H) \sqsubseteq (q^{1/2}, 1]$. The representation Hilbert space is the direct sum $\mathcal{H} = \bigoplus\limits_{n,k=0}^{\infty} \bigoplus\limits_{l=-\infty}^{\infty} \mathcal{H}_{nkl}$, where $\mathcal{H}_{nkl} = \mathcal{K}$. The action of a, b, c, d is given by (78).

The representation $(I)_{H,\epsilon}$ is irreducible if and only if $\mathcal{K} = \mathbb{C}$. Two such representation $(I)_{H,\epsilon}$ and $(I)_{H',\epsilon'}$ are unitarily equivalent if and only if $H = H'$ and $\epsilon = \epsilon'$.

In the rest of this subsection we assume that $\mathcal{K} = \mathbb{C}$ and we study the irreducible representation $(I)_{H,\epsilon}$, $H \in (q^{1/2}, 1]$, more in detail. Since the representation $(I)_{H,\epsilon}$ goes into $(I)_{H,-\epsilon}$ if the generators E, F, K of $\mathcal{U}_q(su_2)$ are replaced by their negatives, we can restrict ourselves to $(I)_{H,1}$. If $H \neq 1$, then the operator K has eigenvalues different from $q^{j/2}$, $j \in \mathbb{Z}$, and hence the corresponding representation of $\mathcal{U}_q(su_2)$ is not integrable.

Fix a unit vector η of $\mathcal{K} = \mathbb{C}$. Let E, F, K, K^{-1} denote the operators of the series $(I)_{H,1}$ defined on the dense domain

$$\mathcal{D}_0 := \text{Lin}\{\eta_{nkl} \,;\, n, k \in \mathbb{N}_0, \, l \in \mathbb{Z}\}$$

of $\mathcal{H}$ and let $\overline{E}, \overline{F}, \overline{K}, \overline{K^{-1}}$ denote their closures. A crucial role plays the vector

$$v_0 := \sum_{n=0}^{\infty} (-q)^n H^{2n} \eta_{n,n,-n}.$$

Lemma 6.1 *For the representation* $(I)_{H,1}$*, the vector* v_0 *belongs to the intersection of domains* $\mathcal{D}(\overline{E}) \cap \mathcal{D}(\overline{F}) \cap \mathcal{D}(\overline{K}) \cap \mathcal{D}(\overline{K^{-1}})$ *and we have* $\overline{E}v_0 = 0, \overline{K}v_0 = Hv_0, \overline{K^{-1}}v_0 = H^{-1}v_0,$

$$\overline{F}v_0 = \lambda^{-1}q^{-1/2}\sum_{n=1}^{\infty}(-1)^n\lambda_n H^{2n}(H - H^{-3})\eta_{n,n-1,-n}.$$

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In particular, $\overline{F}v_0 = 0$ if $H = 1$ and $\overline{F}v_0 \neq 0$ if $H \neq 1$.

Proof. We prove that $v_0 \in \mathcal{D}(\overline{E})$ and $\overline{E}v_0 = 0$. We set

$$v_{tkm} := \sum_{n=0}^{k-1}(-1)^n q^n H^{2n}\eta_{n,n,-n} + \sum_{n=0}^{m}(-1)^{k+n}t^n q^{k+n}H^{2(k+n)}\eta_{k+n,k+n,-k-n}.$$

for $k, m \in \mathbb{N}$, $t \in (0,1)$. Clearly, $v_{tkm} \in \mathcal{D}_0$. By the formula for E we compute

$$q^{1/2}\lambda Ev_{tkm} = \sum_{n=0}^{m-1}\left\{(-1)^{k+n}(1-t)t^n \lambda_{k+n+1}H^{2(k+n)+1}\eta_{k+n,k+n+1,-k-n}\right\}$$
$$+ (-1)^{k+m}t^m \lambda_{k+m+1}H^{2(k+m)+1}\eta_{k+m,k+m+1,-k-m}.$$

Using the facts that t, H, λ_i are in $(0,1]$ we estimate

$$\| v_0 - v_{tkm} \| \leq \sum_{n=k}^{\infty}q^n H^{2n} + \sum_{n=0}^{m}t^n q^{k+n}H^{2(k+n)} \leq 2(1-q)^{-1}q^k,$$

$$q\lambda^2\| Ev_{tkm} \|^2 \leq \sum_{n=0}^{m-1}t^{2k}(1-t)^2 H^{4(k+n)+2} + t^{2m}H^{4(k+m)+2}$$
$$\leq (1-t)(1+t)^{-1} + t^{2m}$$

Let $\varepsilon > 0$. For large $k \in \mathbb{N}$ we have $\| v_0 - v_{tkm} \| < \varepsilon$. Now we choose $t \in (0,1)$ such that $(1-t)(1+t)^{-1} < \varepsilon$. Then we take $m \in \mathbb{N}$ such that $t^{2m} < \varepsilon$. Thus, we have $q\lambda^2\| Ev_{tkm} \|^2 < 2\varepsilon$. This shows that $v_0 \in \mathcal{D}(\overline{E})$ and $\overline{E}v_0 = 0$. The assertion for F follows in a similar manner by a slight modification of the preceding reasoning. Since $K^{\pm 1}v_{tkm} = H^{\pm 1}v_{tkm}$, we conclude that $v_0 \in \mathcal{D}(\overline{K^{\pm 1}})$, $\overline{K}v_0 = Hv_0$ and $\overline{K^{-1}}v_0 = H^{-1}v_0$. $\qquad\square$

If we apply the formula for F formally to the vector $\overline{F}v_0$, we obtain

$$\sum_{n=2}^{\infty}\lambda^{-2}q^{-1}(-1)^n\lambda_{n-1}(\lambda_n - \lambda_{n-1}q^2 H^{-4})(H^2 - H^{-2})(q^{-1}H^2)^n\eta_{n,n-2,-n}.$$

If $H \neq 1$, then $\lim_{n\to\infty}(\lambda_n - \lambda_{n-1}q^2 H^{-4}) = 1 - q^2 H^{-4} > 0$ and $q^{-1}H^2 > 1$ because $H \in (q^{1/2}, 1]$, so this series does not belong to the Hilbert space. By a more

precise argument it can be shown that in the case $H \neq 1$ the vector v_0 is not in the domain of $\overline{F^2}$.

Now suppose that $H = 1$. Then, by Lemma 6.1, $\overline{E}v_0 = \overline{F}v_0 = 0$ and $\overline{K}v_0 = \overline{K^{-1}}v_0 = v_0$. These relations are the key in order to prove that, roughly speaking, the representation $(I)_{1,1}$ is the Heisenberg representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$. In order to do so, we have to pass to a larger domain which contains the vector v_0.

Theorem 6.2 *There is a unique $*$-representation π of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ on the dense domain $\mathcal{D} = \mathcal{O}(SU_q(2))v_0$ of $\mathcal{H}$ such that*

$$\pi(E) \subseteq \overline{E}, \pi(F) \subseteq \overline{F}, \pi(K^{\pm 1}) \subseteq \overline{K^{\pm 1}}$$

and

$$\pi(x) = x, \quad x \in \mathcal{O}(SU_q(2)),$$

where all operators are given by the formulas of $(I)_{1,1}$. The closure of this representation π is unitarily equivalent to the Heisenberg representation π_h of the cross product $*$-algebra $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$.

Proof. Recall that for $f = E$ and $x \in \mathcal{O}(SU_q(2))$ relation (3) of the cross product algebra $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ reads

$$Ex = \langle K^{-1}, x_{(1)} \rangle x_{(2)} E - q^{-1} \langle E, x_{(1)} \rangle x_{(2)} K^{-1}. \tag{84}$$

Since $(I)_{1,1}$ is a representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$, this formula remains valid for the corresponding operators on $\mathcal{D}_0$. By the proof of Lemma 6.1, there exists a sequence of vectors $w_k \in \mathcal{D}_0$, $k \in \mathbb{N}$, such that $Ew_k \to \overline{E}v_0 = 0$ and $K^{-1}w_k \to \overline{K^{-1}}v_0 = v_0$. We apply both sides of (84) to w_k and pass to the limit $k \to \infty$. Since the operators of $\mathcal{O}(SU_q(2))$ are bounded, we obtain
$$\overline{E}xv_0 = -q^{-1}\langle E, x_{(1)} \rangle x_{(2)} v_0 = (x \triangleleft S^{-1}(E))v_0.$$

Similarly, we get
$$\overline{F}xv_0 = (x \triangleleft S^{-1}(F))v_0 \text{ and } \overline{K^{\pm 1}}xv_0 = (x \triangleleft S^{-1}(K^{\pm 1}))v_0.$$

Since $\triangleleft$ is a right action of $\mathcal{U}_q(su_2)$ on $\mathcal{O}(SU_q(2))$, it follows from these formulas that there is a $*$-representation π_0 of $\mathcal{U}_q(su_2)$ on $\mathcal{D}$ such that $\pi_0(f) \subseteq \overline{f}$ for $f = E, F, K, K^{-1}$ and

$$\pi_0(f)xv_0 = (x \triangleleft S^{-1}(f))v_0, \ f \in \mathcal{U}_q(su_2), x \in \mathcal{O}(SU_q(2)). \tag{85}$$

By (85), there is a $*$-representation π of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ such that $\pi(f) = \pi_0(f)$ for $f \in \mathcal{U}_q(su_2)$ and $\pi(x) = x$ for $x \in \mathcal{O}(SU_q(2))$. Since $\pi(f)v_0 = $

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$\varepsilon(f)v_0$ by (85), the linear functional h on $\mathcal{O}(SU_q(2))$ defined by $h(\cdot) = \| v_0 \|^{-2} \langle \pi(\cdot)v_0, v_0 \rangle$ is $\mathcal{U}_q(su_2)$-invariant and satisfies $h(1) = 1$. By Lemma 5.2, h is the Haar state of $\mathcal{O}(SU_q(2))$. By the uniqueness assertion of Proposition 5.3, the closure of π is unitarily equivalent to the Heisenberg representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$. Since the representation $(I)_{1,1}$ is irreducible, it follows from [S], Proposition 8.3.11(i), that $\mathcal{D}$ is dense in $\mathcal{H}$. $\qquad\square$

6.3 The Haar state of $\mathbf{\mathcal{O}(SU_q(2))}$ as a partial quantum trace

In this subsection we use the $*$-representation $(I)_{H,\epsilon}$ of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ to develop another approach to the Haar state h of $\mathcal{O}(SU_q(2))$. We fix a unit vector $\eta \in \mathcal{K}$ and define a linear functional h_0 on $\mathcal{O}(SU_q(2))$ by

$$h_0(x) = (1 - q^2)\sum_{n=0}^{\infty} q^{2n} \langle x\eta_{n,0,-n}, \eta_{n,0,-n} \rangle, \ \ x \in \mathcal{O}(SU_q(2)).$$

Using (78) it is easy to compute $h_0(x)$ on monomials $x = a^i b^r c^s, d^j b^r c^s$ and to see that it coincides with $h(x)$ given by formula (61). Thus, $h_0 = h$. Hence h_0 is $\mathcal{U}_q(su_2)$-invariant because h is so. We will give an independent proof of the $\mathcal{U}_q(su_2)$-invariance of h_0 by using the cross product algebra $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$. From now on we assume that $\mathcal{K} = \mathbb{C}$. Recall that $H \in (q^{1/2}, 1]$ and $\epsilon \in \{1, -1\}$.

Let P be the orthogonal projection of $\mathcal{H}$ on the closure of the subspace

$$\mathcal{D} = \mathrm{Lin}\{e_n := \eta_{n,0,-n} \, ; \, n \in \mathbb{N}_0\}$$

and let $\mathcal{P}$ be the set of operators T defined on $\mathcal{D}$ for which the closure of PTP is of trace class on $\mathcal{H}$. We set

$$\mathrm{Tr}_P \, T := \mathrm{Tr} \, \overline{PTP}, \ \ T \in \mathcal{P}.$$

The main ingredient of our invariance proof is the following partial trace property of the functional Tr_P.

Lemma 6.3 *For $x \in \mathcal{O}(SU_q(2))$ and $f \in \{EK^{-1}, FK^{-1}, K^{-1}\}$, we have $fx, xf \in \mathcal{P}$ and $\mathrm{Tr}_P \, fx = \mathrm{Tr}_P \, xf$.*

Proof. Since $\{e_n; n \in \mathbb{N}_0\}$ is a orthonormal basis of $P\mathcal{H}$, we have

$$\mathrm{Tr}_P \, T = \sum_{n=0}^{\infty} \langle Te_n, e_n \rangle \, . \tag{86}$$

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Set $a^{\#k} := a^k, a^{\#(-k)} := a^{*k}, c^{\#k} := c^k, c^{\#(-k)} := c^{*k}$ for $k \in \mathbb{N}_0$. It suffices to prove the assertion for $x = a^{\#k}c^{\#l}(c^*c)^j$, $k, l \in \mathbb{Z}, j \in \mathbb{N}_0$, because $\mathcal{O}(SU_q(2))$ is the linear span of these elements. First let $f = K^{-1}$. If $k \neq 0$ or $l \neq 0$, then $\mathrm{Tr}_P K^{-1}x = \mathrm{Tr}_P xK^{-1} = 0$ since $K^{-1}x\eta_{n,0,-n}, xK^{-1}\eta_{n,0,-n} \in \mathcal{H}_{n-k,0,-n-l}$. If $k = l = 0$, then $PK^{-1}xPe_n = PxK^{-1}Pe_n = q^{(2j+1)n}e_n$. Hence the assertion holds for $f = K^{-1}$.

Let $f = EK^{-1}$. The operator EK^{-1} on $\mathcal{D}_0$ can by written as $EK^{-1} = S + q^{-3/2}\lambda^{-1}c^{-1}aK^{-2}$, where S maps $e_n = \eta_{n,0,-n}$ on multiples of $\eta_{n,1,-n}$. Hence we have $PSxP = PxSP = 0$. Thus it suffices to prove the assertion for $f = c^{-1}aK^{-2}$. If $k \neq -1$ or $l \neq 1$, then $\mathrm{Tr}_P c^{-1}aK^{-2}x = \mathrm{Tr}_P xc^{-1}aK^{-2} = 0$ since $c^{-1}aK^{-2}x\eta_{n,0,-n}, xc^{-1}aK^{-2}\eta_{n,0,-n} \in \mathcal{H}_{n-k-1,0,-n-l+1}$. Now let $k = -1, l = 1$. Then,

$$c^{-1}aK^{-2}x = q^2(c^*c)^jK^{-2} - q^4(c^*c)^{j+1}K^{-2},$$
$$xc^{-1}aK^{-2} = q^{-2j}(c^*c)^jK^{-2} - q^{-2j}(c^*c)^{j+1}K^{-2}.$$

Since $\mathrm{Tr}_P (c^*c)^m K^{-2} = H^{-2}(1-q^{2m+2})^{-1}$ by (86), it follows from these identities that $\mathrm{Tr}_P c^{-1}aK^{-2}x = \mathrm{Tr}_P xc^{-1}aK^{-2}$.
The proof for $f = FK^{-1}$ is similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 6.4 *The functional $h_0(x) = (1-q^2)H^2\mathrm{Tr}_P K^{-2}x$, $x \in \mathcal{O}(SU_q(2))$, is $\mathcal{U}_q(su_2)$-invariant and satisfies $h_0(1) = 1$. In particular, it is the Haar state of $\mathcal{O}(SU_q(2))$.*

Proof. Since $K^{-2}e_n = q^{2n}H^{-2}e_n$, $h_0(x) = (1-q^2)H^2\mathrm{Tr}_P K^{-2}x$ by (86). Let $x \in \mathcal{O}(SU_q(2))$. There is an element $y \in \mathcal{O}(SU_q(2))$ such that $xK = Ky$. Then we have

$$h_0(x \triangleleft E) = h_0(ad_R(E)x) = h_0(KxE - qExK)$$
$$= q^{-1}(h_0(K^2yEK^{-1}) - h_0(K^2EK^{-1}y))$$
$$= q^{-1}(1-q^2)H^2(\mathrm{Tr}_P yEK^{-1} - \mathrm{Tr}_P EK^{-1}y) = 0$$

by Lemma 6.3. Similarly, $h_0(x \triangleleft F) = 0$ and $h_0(x \triangleleft K^{\pm 1}) = h_0(x)$, so h_0 is $\mathcal{U}_q(su_2)$-invariant. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 6.5 In fact, if $\{k_n\}$ and $\{l_n\}$ are arbitrary sequences from $\mathbb{N}_0$ and $\mathbb{Z}$, respectively, then the Haar state h of $\mathcal{O}(SU_q(2))$ can be written as

$$h(x) = (1-q^2)\sum_{n=0}^{\infty} q^{2n}\langle x\eta_{n,k_n,l_n}, \eta_{n,k_n,l_n}\rangle.$$

6.4 Representations of the ∗-algebra $\mathcal{U}_{\mathbf{q}}(\mathbf{gl_2}) \ltimes \hat{\mathcal{O}}(\mathbb{C}^2_{\mathbf{q}})$

We argue similarly as in the preceding two subsections and begin with a ∗-representation of the ∗-algebra $\mathcal{U}_q(gl_2) \ltimes \hat{\mathcal{O}}(\mathbb{C}^2_q)$ such that its restriction to $\hat{\mathcal{O}}(\mathbb{C}^2_q)$ is admissible and hence of the form described in 4.2.

Since $\mathcal{G}$ is the kernel of z_1 and z_1^*, it follows from (28), (30) and (32) that $\mathcal{G}$ is also the kernel of z_2 and z_2^* and that the representation leaves $\mathcal{G}$ invariant. On $\mathcal{G}$ we have $z_1 = z_1^* = z_2 = z_2^* = 0$ and an arbitrary ∗-representation of $\mathcal{U}_q(gl_2)$. Such a ∗-representation of $\mathcal{U}_q(gl_2) \ltimes \hat{\mathcal{O}}(\mathbb{C}^2_q)$ will be called *trivial*.

From now on we assume that $\mathcal{G} = \{0\}$. Since $wAw^* = A$ by assumption, the operator $N := wA$ is normal. The only differences to the ∗-algebra $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$ are the strictly positive operator A and the additional generator L which satisfies (33) and belongs to the center of the algebra $\mathcal{U}_q(gl_2)$. From the relation $z_2^* L = pLz_2^*$ and the fact that L is unitary it follows that L leaves each space $\mathcal{H}_n$ invariant. Hence there are unitaries L_n on $\mathcal{H}_n$ such that $L\eta_n = L_n\eta_n$. From $z_1^* z_1 L = p^2 L z_1^* z_1$ we get $A^2 L_n = p^2 L_n A^2$ and so $L_n^* A L_n = pA$. Combining the latter with the relations $z_2 L = pLz_2$ and $z_1 L = pLz_1$ and using the fact that ker $A = \{0\}$ we derive that $L_n = L_0$ for all n and that $wL_0w^* = L_0$. Since L commutes with E, L_0 commutes with T_0 and K_0. The relation $z_1 E = q^{-1/2}Ez_1$ implies $AT_0 = T_0A$ and $AK_0 = K_0A$. Summarizing, it follows that the operators E, F, K, L act as

$$E\eta_n = q^{-n/2}T_0\eta_n + \lambda^{-1}\lambda_n q^{-n/2}K_0^{-1}w\eta_{n-1}, \tag{87}$$

$$F\eta_n = q^{-n/2}T_0^*\eta_n + \lambda^{-1}\lambda_{n+1}q^{-(n+1)/2}w^*K_0^{-1}\eta_{n+1}, \tag{88}$$

$$K\eta_n = q^{-n/2}K_0\eta_n, \; L\eta_n = L_0\eta_n, \tag{89}$$

where w, L_0, T_0 and K_0 are operators on the Hilbert space $\mathcal{H}_0$ such that w and L_0 are unitaries, K_0 is invertible and self-adjoint, and the conditions

$$wK_0w^* = q^{-1/2}K_0, \; K_0T_0 = qT_0K_0, \; wT_0w^* = q^{-1/2}T_0, \; wL_0 = L_0w, \tag{90}$$

$$T_0T_0^* - T_0^*T_0 = \lambda^{-1}K_0^2, \tag{91}$$

$$wAw^* = A, \; AT_0 = T_0A, \; AK_0 = K_0A, \tag{92}$$

$$L_0AL_0^* = p^{-1}A, \; L_0K_0L_0^* = K_0, \; L_0T_0L_0^* = T_0, \tag{93}$$

hold. Here A is a strictly positive self-adjoint operator on $\mathcal{H}_0$. Conversely, if operators w, A, L_0, T_0 and K_0 on a Hilbert space $\mathcal{H}_0$ are given satisfying the preceding conditions (say on an invariant dense domain $\mathcal{D}_0$ of $\mathcal{H}_0$

such that $K_0 \mathcal{D}_0 = \mathcal{D}_0$), then above formulas define a $*$-representation of $\mathcal{U}_q(gl_2) \ltimes \hat{\mathcal{O}}(\mathbb{C}_q^2)$.

Note that the operators T_0, K_0 and w satisfy precisely the same relations as in case of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(SU_q(2))$. Proceeding as in 6.2 we obtain the following list of non-trivial $*$-representations of $\mathcal{U}_q(gl_2) \ltimes \hat{\mathcal{O}}(\mathbb{C}_q^2)$:

$(I)_{H,B,\epsilon}$:

$E\eta_{nklj} = -q^{-(n-l+1)/2}\lambda_{k+1}\lambda^{-1}\epsilon H \eta_{n,k+1,l,j} + q^{-(n+2k+l+1)/2}\lambda_n\lambda^{-1}\epsilon H^{-1}\eta_{n-1,k,l+1,j}$,

$F\eta_{nklj} = -q^{-(n-l+1)/2}\lambda_k\lambda^{-1}\epsilon H \eta_{n,k-1,l,j} + q^{-(n+2k+l+1)/2}\lambda_{n+1}\lambda^{-1}\epsilon H^{-1}\eta_{n+1,k,l-1,j}$,

$K\eta_{nklj} = q^{-(n-2k-l)/2}\epsilon H \eta_{nklj}$, $\quad L\eta_{nklj} = \eta_{nkl,j+1}$,

$z_1\eta_{nklj} = q^{n+1}p^j B\eta_{nk,l+1,j}$, $\quad z_1^*\eta_{nklj} = q^{n+1}p^j B\eta_{nk,l-1,j}$,

$z_2\eta_{nklj} = \lambda_{n+1}p^j B\eta_{n+1,klj}$, $\quad z_2^*\eta_{nklj} = \lambda_n p^j B\eta_{n-1,klj}$,

where $\epsilon \in \{-1, 1\}$. The parameters H and B denote self-adjoint operators acting on a Hilbert space $\mathcal{K}$ such that $\sigma(H) \sqsubseteq (q^{1/2}, 1]$ and $\sigma(B) \sqsubseteq (p, 1]$ if $p < 1$ resp. $\sigma(B) \sqsubseteq [1, p)$ if $p > 1$. The underlying Hilbert space is the direct sum $\mathcal{H} = \bigoplus_{n,k=0}^{\infty} \bigoplus_{l,j=-\infty}^{\infty} \mathcal{H}_{nklj}$, where $\mathcal{H}_{nklj} = \mathcal{K}$. Representations corresponding to different sets $\{H, B, \epsilon\}$ of parameters are not unitarily equivalent. A representation of this series is irreducible if and only if $\mathcal{K} = \mathbb{C}$.

The representations of the $*$-subalgebra $\mathcal{U}_q(su_2) \ltimes \hat{\mathcal{O}}(\mathbb{C}_q^2)$ are obtained by the same formulas as above when the last index j, the constants p^j, and the operator L are omitted. In this case B is strictly positive.

6.5 Representations of the $*$-algebra $\mathcal{U}_\mathbf{q}(\mathbf{gl_2}) \ltimes \mathcal{O}(\mathbb{R}_\mathbf{q}^3)$

Suppose we have a $*$-representation of the $*$-algebra $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$. From the defining relations of the algebra $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$ it follows easily that the subspace $\mathcal{K}_0 := \ker x_2$ is invariant under all generators and that $x_1 = x_3 = 0$ on $\mathcal{K}_0$. On $\mathcal{K}_0$ we can have an arbitrary $*$-representation of $\mathcal{U}_q(gl_2)$. Such a $*$-representation of $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$ is called *trivial*.

1. Step: From now we assume that $\ker x_2 = \{0\}$ and that the restriction of the $*$-representation to $\mathcal{O}(\mathbb{R}_q^3)$ is admissible, that is, it is of the form given in 4.2. Since $\ker x_1^n = \mathcal{H}_0 + \cdots + \mathcal{H}_{n-1}$, the relation $x_1 K = q^{-1} K x_1$ implies that K leaves the subspace $\mathcal{H}_0 + \cdots + \mathcal{H}_{n-1}$ invariant. Since K is symmetric, K leaves $\mathcal{H}_n$ invariant so that there are operators K_n on $\mathcal{H}_0$ such that $K\eta_n = K_n\eta_n$. The hermitian elements K and $\mathcal{Q}_q^2$ of the algebra $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$

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commute. We assume that the corresponding self-adjoint operators *strongly* commute. This implies that $AK_n = K_n A$ on $\mathcal{H}_0$. Combining the latter with the relation $x_3 K = qK x_3$, we derive $K_n = q^{-n} K_0$. A slight modification of this reasoning shows that the operator L acts as $L\eta_n = L_0 \eta_n$, where L_0 is a unitary operator on $\mathcal{H}_0$ such that $L_0^* A L_0 = p^2 A$. From (45) and (46) it follows that $wK_0 = K_0 w$ and $wL_0 = L_0 w$.

Using the relations $x_1 E = q^{-1} E x_1$ and $x_3 E = qE x_3 + q\gamma K^{-2} x_2$ it can be shown by induction on n that E maps $\mathcal{H}_n$ into $\mathcal{H}_{n-1} + \mathcal{H}_n$. Write $E\eta_n = T_n \eta_n + S_n \eta_{n-1}$, where T_n and S_n are operators on $\mathcal{H}_0$. Since E commutes with $\mathcal{Q}_q^2$, we get $A^2 S_n = S_n A^2$ and $A^2 T_n = T_n A^2$. We assume that $AT_n = T_n A$. From $A^2 S_n = S_n A^2$ and the relation $x_1 x_3 E = E x_1 x_3 + q^2 \gamma K^{-1} x_1 x_2$ we obtain by comparing coefficients

$$S_n = q^{-1/2} q^{-n} \lambda^{-1} \lambda_{2n} K_0^{-1} w.$$

From $x_1 E = q^{-1} E x_1$ it follows that $T_n = q^{-n} T_0$. The relations $KE = qEK$, $LE = EL, LK = KL$ and $x_2 E = E x_2 - q\gamma K^{-1} x_1$ give $K_0 T_0 = qT_0 K_0, L_0 T_0 = T_0 L_0, L_0 K_0 = K_0 L_0$ and $wT_0 = T_0 w$, respectively. Inserting the expressions of E and K into the equation $EE^* - E^* E = \lambda^{-1}(K^2 - K^{-2})$, we derive $T_0 T_0^* - T_0^* T_0 = \lambda^{-1}(K_0^2 + q^{-2} K_0^{-2})$.

We now summarize the preceding. The operators E, F, K and L act as

$$E\eta_n = q^{-n} T_0 \eta_n + q^{-1/2} q^{-n} \lambda^{-1} \lambda_{2n} K_0^{-1} w \eta_{n-1}, \tag{94}$$

$$F\eta_n = q^{-n} T_0^* \eta_n + q^{-3/2} q^{-n} \lambda^{-1} \lambda_{2(n+1)} K_0^{-1} w \eta_{n+1}, \tag{95}$$

$$K\eta_n = q^{-n} K_0 \eta_n, \quad L\eta_n = L_0 \eta_n, \tag{96}$$

where the operators T_0, K_0, L_0 and A satisfy the consistency conditions

$$T_0^* T_0 - T_0 T_0^* = -\lambda^{-1}(K_0^2 + q^{-2} K_0^{-2}), \tag{97}$$

$$K_0 T_0 = qT_0 K_0, L_0 T_0 = T_0 L_0, L_0 K_0 = K_0 L_0, \tag{98}$$

$$AT_0 = T_0 A, AK_0 = K_0 A, L_0 A L_0^* = p^{-2} A, \tag{99}$$

$$wT_0 = T_0 w, wK_0 = K_0 w, wL_0 = L_0 w, wA = Aw. \tag{100}$$

Conversely, if we have two self-adjoint operators A, K_0, a unitary operator L_0, a self-adjoint unitary w and an operator T_0 on $\mathcal{H}_0$ satisfying (97)–(100), then the formulas (54)–(55) and (94)–(96) define a $*$-representation of $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$.

2. Step: Now we analyze relations (97)–(99). Let $T_0 = v|T_0|$ be the polar decomposition of the closed operator T_0. From (97) we conclude that

$\ker(T_0^* T_0) = \{0\}$ and so $\ker v = \ker |T_0| = \{0\}$. Hence v is an isometry. From $K_0 T_0 = q T_0 K_0$ we get $K_0 T_0^* T_0 = T_0^* T_0 K_0$. We assume that K_0 and $T_0^* T_0$ are strongly commuting self-adjoint operators. Then $K_0 |T_0| = |T_0| K_0$ and hence $q v K_0 = K_0 v$. Let $v = v_u \oplus v_s$ on $\mathcal{H} = \mathcal{H}^u \oplus \mathcal{H}^s$ be the Wold decomposition of the isometry v from Lemma 4.1.

Since $v^* v = 1$ and $T_0 T_0^* = v T_0^* T_0 v^*$, (97) yields

$$T_0^* T_0 v^n = v T_0^* T_0 v^{n-1} - \lambda^{-1}(K_0^2 + q^{-2} K_0^{-2}) v^n, \ n \in \mathbb{N} .$$

Since $q v K_0 = K_0 v$, by Lemma 4.2 $\mathcal{H}^u$ reduces K_0. Using this fact we deduce from the preceding identity that $T_0^* T_0$ leaves $\mathcal{H}^u = \cap_{n=0}^\infty v^n \mathcal{H}$ invariant. We assume that $\mathcal{H}^u$ is even reducing for $T_0^* T_0$. Then $\mathcal{H}^u$ is also reducing for T_0 and T_0^*. Let us denote the restrictions of K_0, T_0 and T_0^* by the same symbols. For the unitary part v_u of v it follows from (97) that

$$0 < v_u^n |T_0|^2 v_u^{*n} = |T_0|^2 + \lambda^{-1} \sum_{k=0}^{n-1} (q^{-2k} K_0^2 + q^{2(k-1)} K_0^{-2})$$

$$= |T_0|^2 - q\lambda^{-2}(1 - q^{2n})(q^{-2n} K_0^2 + K_0^{-2})$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ we conclude that the latter is only possible when $\mathcal{H}^u = \{0\}$. That is, we have $v = v_s$ and $\mathcal{H} = \mathcal{H}^s$.

The operator relation $q v K_0 = K_0 v$ is treated by Lemma 4.2(ii). Then there is an invertible self-adjoint operator H on $\mathcal{H}_{00}$ such that $v\zeta_k = \zeta_{k+1}$, $K_0 \zeta_k = q^k H \zeta_k$ on $\mathcal{H}_0 = \oplus_{k=0}^\infty \mathcal{H}_{0k}$, $\mathcal{H}_{0k} = \mathcal{H}_{00}$. Since $\ker T_0^* = \ker v^* = \mathcal{H}_{00}$, Equation (97) gives $|T_0|^2 \zeta_0 = -\lambda^{-1}(H^2 + q^{-2} H^{-2})\zeta_0$. Using (97) and the fact that $K_0 v = q v K_0$ we compute

$$|T_0|^2 \zeta_k = |T_0|^2 v^k \zeta_0 = v^k (|T_0|^2 - \lambda^{-1} \sum_{l=1}^k (q^{2l} K_0^2 + q^{-2(l+1)} K_0^{-2}))\zeta_0$$

$$= q^{-1} \lambda^{-2}(1 - q^{2(k+1)})(H_0^2 + q^{-2(k+1)} H_0^{-2})\zeta_k ,$$

and hence

$$|T_0|\zeta_k = -q^{-1/2}\lambda^{-1}\lambda_{k+1}(H_0^2 + q^{-2(k+1)} H_0^{-2})^{1/2}\zeta_k .$$

From $L_0 T_0 = T_0 L_0, L_0 K_0 = K_0 L_0, A T_0 = T_0 A, A K_0 = K_0 A$ and $L_0 A L_0^* = p^{-2} A$ it follows by similar reasoning as used in Section 6.1 that there are a unitary L_{00} and a self-adjoint operator A_0 on the Hilbert space $\mathcal{H}_{00}$ satisfying $L_{00} H = H L_{00}, A_0 H = H A_0$ and $L_{00} A_0 L_{00}^* = p^{-2} A_0$ such that $L_0 \zeta_k =$

$L_{00}\zeta_k$, $A\zeta_k = A_0\zeta_k$. The representations of the relation $L_{00}A_0L_{00}^* = p^{-2}A_0$ are taken from Lemma 4.2(i). The relations (100) are treated similarly.

Carrying out the details we obtain the following list of non-trivial $*$-representations of $\mathcal{U}_q(gl_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$:

$$(I)_{A,H,w}: \quad E\eta_{nkl} = -q^{-n-1/2}\lambda^{-1}\lambda_{k+1}(H^2 + q^{-2(k+1)}H^{-2})^{1/2}\eta_{n,k+1,l}$$
$$+ q^{-n-k-1/2}\lambda^{-1}\lambda_{2n}H^{-1}w\eta_{n-1,kl},$$
$$F\eta_{nkl} = -q^{-n-1/2}\lambda^{-1}\lambda_k(H^2 + q^{-2k}H^{-2})^{1/2}\eta_{n,k-1,l}$$
$$+ q^{-n-k-3/2}\lambda^{-1}\lambda_{2(n+1)}H^{-1}w\eta_{n+1,kl},$$
$$K\eta_{nkl} = q^{-n+k}H\eta_{nkl}, \quad L\eta_{nkl} = \eta_{nk,l+1},$$
$$x_1\eta_{nkl} = (1+q^2)^{-1/2}\lambda_{2n}p^{2l}A\eta_{n-1,kl}, \quad x_2\eta_{nkl} = q^{2n+1}p^{2l}Aw\eta_{nkl},$$
$$x_3\eta_{nkl} = q(1+q^2)^{-1/2}\lambda_{2(n+1)}p^{2l}A\eta_{n+1,kl},$$

where w, A and H are commuting self-adjoint operators on a Hilbert space $\mathcal{K}$ such that w is unitary, H is invertible, $\sigma(A) \subseteq (p^2, 1]$ if $p<1$, and $\sigma(A) \subseteq (p^{-2}, 1]$ if $p>1$. The representation space is $\mathcal{H} = \overset{\infty}{\underset{n,k=0}{\oplus}} \overset{\infty}{\underset{l=-\infty}{\oplus}} \mathcal{H}_{nkl}$, $\mathcal{H}_{nkl} = \mathcal{K}$.

If we omit the operator L, the last index l, the constants p^{2l}, and assume that A is a strictly self-adjoint operator, then the above formulas describe non-trivial $*$-representations of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$. If we set in addition $A = 1$ and rename x_i by y_i, then we obtain $*$-representations of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(S_q^2)$.

The cases $H = 1$ and $H = q^{1/2}$ are of particular interest. Then the above representations of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$ are determined by the following formulas:

$$(I)_{A,1,w}: \quad E\eta_{nk} = q^{-n-k-1/2}\lambda^{-1}(-q^{-1}\lambda_{2(k+1)}\eta_{n,k+1} + \lambda_{2n}w\eta_{n-1,k}),$$
$$F\eta_{nk} = q^{-n-k-1/2}\lambda^{-1}(-\lambda_{2k}\eta_{n,k-1} + q^{-1}\lambda_{2(n+1)}w\eta_{n+1,k}),$$
$$K\eta_{nk} = q^{-n+k}\eta_{nk},$$
$$(I)_{A,q^{1/2},w}: \quad E\eta_{nk} = q^{-n-k-1}\lambda^{-1}(-q^{-1}\lambda_{k+1}\alpha_{k+2}\eta_{n,k+1} + \lambda_n\alpha_n w\eta_{n-1,k}),$$
$$F\eta_{nk} = q^{-n-k-1}\lambda^{-1}(-\lambda_k\alpha_{k+1}\eta_{n,k-1} + q^{-1}\lambda_{n+1}\alpha_{n+1}w\eta_{n+1,k}),$$
$$K\eta_{nk} = q^{-n+k+1/2}\eta_{nk},$$

where $\alpha_k := (1 + q^{2k})^{1/2}$. In both cases the operators x_1, x_2 and x_3 act as

$$x_1\eta_{nk} = (1+q^2)^{-1/2}\lambda_{2n}A\eta_{n-1,k}, \quad x_2\eta_{nk} = q^{2n+1}Aw\eta_{nk},$$

$$x_3\eta_{nk} = q(1+q^2)^{-1/2}\lambda_{2(n+1)}A\eta_{n+1,k}.$$

Here A is a strictly positive self-adjoint operator and w is a self-adjoint unitary operator on a Hilbert space $\mathcal{K}$ such that $wAw^* = A$. The representation space is $\mathcal{H} = \bigoplus_{n,k=0}^{\infty} \mathcal{H}_{nk}$, $\mathcal{H}_{nk} = \mathcal{K}$.

In the case $\mathcal{K} = \mathbb{C}$, $H=1$ the representations $(I)_{A,1,w}$ have been constructed in [F] and [CW]. The formulas in [CW] can be obtained as follows. First we replace q by q^{-1}. Then the algebra in [CW] becomes a $*$-subalgebra of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$ by setting $X^+ = -x_1$, $X^- = x_3$, $X^3 = x_2$, $T^+ = -q^{1/2}KE$, $T^- = -q^{-1/2}KF$, and $T^3 = \lambda^{-1}(K^4-1)$. The representation in [CW] is unitarily equivalent to $(I)_{A,1,w}$ with $A = q^{-1-2M}|z_0|$ and $w = \operatorname{sign} z_0$ by the isomorphism $\eta_{nk} = (-1)^k|M, -n+M, k+n\rangle$ if $z_0 > 0$ and $\eta_{nk} = (-1)^{n+k}|M, -n+M, k-n\rangle$ if $z_0 < 0$.

Next we suppose that $\mathcal{K}=\mathbb{C}$, $H=q^{1/2}$, $w=\pm 1$ and $A > 0$. Set $\eta:=1$ and

$$v_{1/2} := \sum_{n=0}^{\infty} w^n q^n \alpha_{n+1} \eta_{n,n}.$$

Then we have $v_{-1/2}:=\overline{F}v_{1/2} \neq 0$, $\overline{E}v_{1/2}=0$, $\overline{F}v_{-1/2}=0$, $\overline{K}v_{\pm 1/2}=q^{\pm 1/2}v_{\pm 1/2}$ and $\overline{K^{-1}}v_{\pm 1/2}=q^{\mp 1/2}v_{\pm 1/2}$. Proceeding similarly as in Theorem 6.2 it follows that $(I)_{A,q^{1/2},w}$ leads to an irreducible closed $*$-representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(\mathbb{R}_q^3)$ such that its restriction to $\mathcal{U}_q(su(2))$ is the direct sum of all representations $T_{l+1/2}$ with $l \in \mathbb{N}_0$. In particular, $(I)_{1,q^{1/2},1}$ and $(I)_{1,q^{1/2},-1}$ give inequivalent closed irreducible $*$-representations of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(S_q^2)$ such that their restrictions to $\mathcal{U}_q(su_2)$ are integrable. Obviously, both representations are not equivalent to the Heisenberg representation.

We now close the link between the approaches in Sections 5 and 6 by describing the Heisenberg representation of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(S_q^2)$ in terms of the representation $(I)_{1,1,w}$. Let $\mathcal{K} := \mathbb{C}^2$ and $\zeta := 2^{-1/2}(1,1) \in \mathbb{C}^2$. Let w be the diagonal matrix with diagonal entries 1 and -1. Then, $w^n\zeta = 2^{-1/2}(1,(-1)^n)$. Put

$$v_0 := (1 - q^2)^{1/2} \sum_{n=0}^{\infty} q^n w^n \zeta_{n,n}.$$

From Theorem 6.6 below it follows that $h(x) := \langle xv_0, v_0\rangle$, $x \in \mathcal{O}(S_q^2)$, is the unique $\mathcal{U}_q(su_2)$-invariant state on $\mathcal{O}(S_q^2)$.

Theorem 6.6 *There is a unique $*$-representation π of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(S_q^2)$ on the domain $\mathcal{D} = \mathcal{O}(S_q^2)v_0$ such that $\pi(x) = x, x \in \mathcal{O}(SU_q(2))$, and*

$$\pi(E) \subseteq F^*, \pi(F) \subseteq E^*, \pi(K^{\pm 1}) \subseteq (K^{\pm 1})^*,$$

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where all operators are given by $(I)_{1,1,w}$. *The closure of the representation* π *is unitarily equivalent to the Heisenberg representation* π_h *of* $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(S_q^2)$.

Proof. Since the proof is similar to the proof of Theorem 6.2, we sketch only the necessary modifications. Put $y_1^{\#n} := y_1^n$ and $y_1^{\#-n} := y_1^{*n}$ for $n \in \mathbb{N}$. Let $f = E, F, K, K^{-1}$. The crucial step of the proof is to show that

$$\langle y_1^{\#r} y_2^s v_0, f \eta_{nk} \rangle = \langle ((y_1^{\#r} y_2^s) \triangleleft S^{-1}(f^*)) v_0, \eta_{nk} \rangle \tag{101}$$

for $s, n, k, \in \mathbb{N}_0$ and $r \in \mathbb{Z}$. The verification of (101) is straightforward. By (101), for any $x \in \mathcal{O}(SU_q(2))$ the vector xv_0 is in the domain of the adjoint operator f^* of f and f^* acts on xv_0 by $f^*(xv_0) = (x \triangleleft S^{-1}(f^*))v_0$. Hence it follows that there is a $*$-representation π of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(S_q^2)$ such that $\pi(x) = x$ and $\pi(f)(xv_0) = (x \triangleleft S^{-1}(f))v_0$ for all $x \in \mathcal{O}(SU_q(2))$ and $f \in \mathcal{U}_q(su_2)$. Since $\pi(f)v_0 = \varepsilon(f)v_0$ and $\| v_0 \| = 1$, h is a $\mathcal{U}_q(su_2)$-invariant state. Hence the closure of π is unitarily equivalent to the Heisenberg representation π_h of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(S_q^2)$. $\qquad\square$

Let π_0 denote the $*$-representation $(I)_{1,1,w}$ of $\mathcal{U}_q(su_2) \ltimes \mathcal{O}(S_q^2)$ on the domain $\mathcal{D}_0 := \text{Lin}\{\eta_{nk}; n, k \in \mathbb{N}_0\}$ and let π_0^* be its adjoint representation ([S], Definition 8.1.4). Then π is just the restriction of π_0^* to the subdomain $\mathcal{D} = \mathcal{O}(S_q^2)v_0$. Since the adjoint representation is not a $*$-representation in general, the fact that π is $*$-preserving has to be proved. In fact, this was done by verifying formula (101).

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